Fall 2021 ME424 Modern Control and Estimation

Linear Algebra Review: Part III
Geometry ~ linear algebra

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Outline

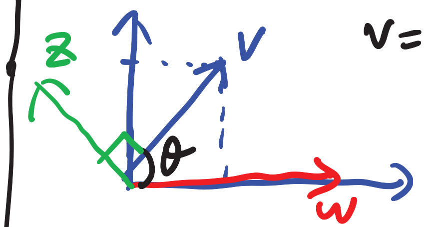
- Inner product
- Projection
- Geometric sets
  - Lines
  - Convex set and cone
  - Hyperplanes
  - Polytopes
  - Ball, ellipsoid

Inner Product

- Inner product of vectors in R^n: $< v, w > \overset{\Delta}{=} v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$

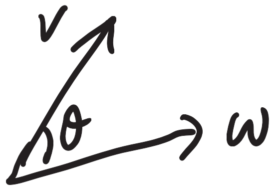
$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \qquad = \sum_{i=1}^{n} v_i w_i = v^T w$$

- Norm of $v \in R^n$: $\|v\| = \sqrt{<v, v>} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$



$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

- Angle between $v, w \in R^n$: $\cos\theta = \dfrac{<v, w>}{\|v\| \|w\|}$



$$\cos\theta = \frac{<v, w>}{\|v\| \|w\|} = \frac{2}{\sqrt{2} \cdot 2} = \frac{\sqrt{2}}{2}$$

$z = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad v^T z = 0$

- Orthogonality:

$v \perp w \Rightarrow \cos\theta = 0 \Rightarrow <v, w> = v^T w = 0$

Inner Product

- General inner product $<\cdot,\cdot>: V \times V \to R$
  - maps each pair in a vector space to a scaler
  - satisfies several key properties: linearity, conjugate, positive definiteness …

$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{1000} \end{bmatrix}$ $g = \begin{bmatrix} g_1 \\ \vdots \\ g_{1000} \end{bmatrix}$

vector space for which inner product can be defined is called inner product space

- Inner product of matrices in $R^{m\times n}$: $< A, B > \triangleq tr(A^T B) = tr(AB^T)$

$m\times n$ matrices

$tr(\cdot)$ = sum of diagonals $= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$

eg: $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

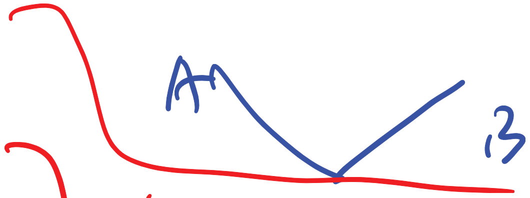
$<A,B> = tr(A^T B) = tr\left(\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = tr\left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\right) = -2$, $\angle(A,B) > 90°$

- Inner product of two functions f, g on interval $[a, b]$:

$\begin{cases} f: [a,b] \to \mathbb{R} \\ g: [a,b] \to \mathbb{R} \end{cases}$

$<f,g> \triangleq \int_a^b f(x)g(x)dx$

or $\triangleq \frac{1}{|b-a|}\int_a^b f(x)g(x)dx$ (another choice)

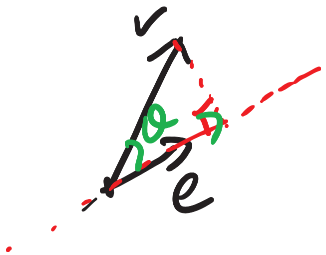


both valid inner product definitions

however, yields different geometry.

Projection

- Projection of $v \in R^n$ along direction e :



suppose $\|e\|=1$, unit vector

$$Proj_e(v) = (\|v\| \cos\theta) \cdot e$$

$$= \frac{\|v\| \langle v, e \rangle}{\|v\| \cdot 1} \cdot e$$

$$= \langle v, e \rangle \cdot e$$

If $\|e\| \neq 1$, we can normalize it

$$Proj_e(v) = \langle v, \frac{e}{\|e\|} \rangle \cdot \frac{e}{\|e\|}$$

- $\{e_1, \dots, e_k\}$ be orthonormal basis of vector space V, then any $v \in V$,

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k$$

① orthonormal $\Rightarrow \begin{cases} e_i^T e_j = 0, & i \neq j \\ e_i^T e_j = 1, & i = j \end{cases}$

$e_i^T e_j$: $\|e_i\|$

② $\dim(V) = k$

③ any $v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k$

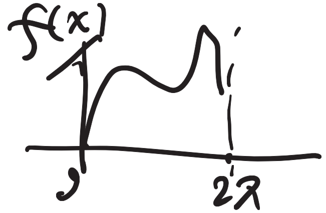
$$\alpha_i = \langle v, e_i \rangle$$

why? :

$$\langle v, e_i \rangle = \langle \alpha_1 e_1, e_i \rangle + \cdots + \langle \alpha_k e_k, e_i \rangle$$

$$= \langle \alpha_i e_i, e_i \rangle = \alpha_i$$

Projection



- **Fourier series**: Consider a vector space of periodic functions:

$$V = \{\text{integrable functions over } [0,2\pi)\}$$

$f \in V \Rightarrow f$ integrable func on $[0,2\pi)$

$\int_0^{2\pi} \|f\|^2 dx < \infty$

- Inner product: $< f, g > = \int_0^{2\pi} f(x)g(x)dx$

- Basis: $B = \{1, \cos x, \sin(x), \cos(2x), \sin(2x), \dots\}$

$\phi_1(x) = 1$, $\phi_2(x)$, $\phi_3(x)$ ...

"ϕ_i" are orthonal:

$$<\phi_i, \phi_j> = \begin{cases} 0, & \text{if } i \neq j \\ \pi, & \text{if } i = j \neq 1 \\ 2\pi, & \text{if } i = j = 1 \end{cases}$$

e.g. $\int_0^{2\pi} \sin^2(x)\, dx = \pi$

- $f \in V$, then

$$f = <f, \frac{\phi_1}{\|\phi_1\|}> \frac{\phi_1}{\|\phi_1\|} + <f, \frac{\phi_2}{\|\phi_2\|}> \frac{\phi_2}{\|\phi_2\|} + \cdots$$

Fourier expansion

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots$$

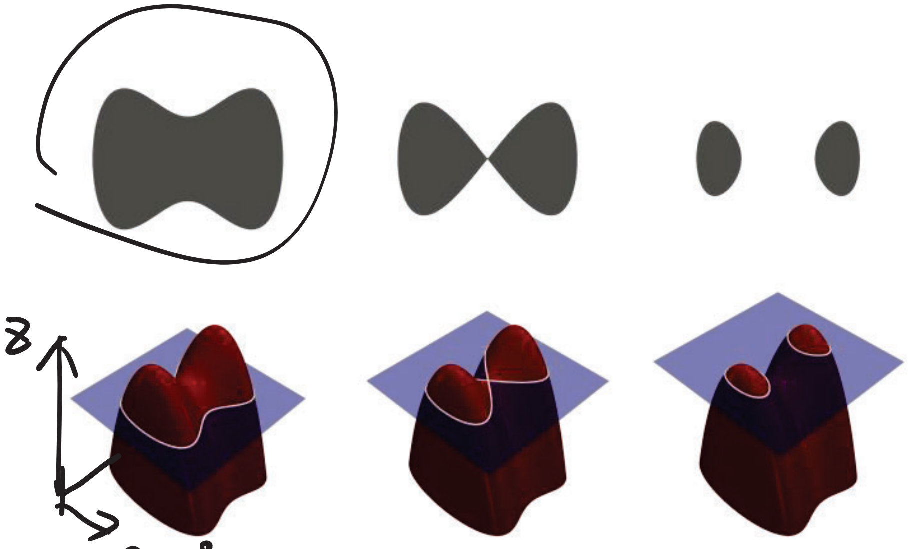
$$\left(\int_0^{2\pi} \frac{f(x) \cdot \cos x}{\sqrt{\pi}} dx\right) \frac{\cos x}{\sqrt{\pi}} = \left(\frac{1}{\pi} \int_0^{2\pi} f(x) \cos(x)\, dx\right) \cos x$$

a_1

Representation of Geometric Objects / sets

- Implicit representation via (sub)-level sets:

$$\{x \in R^n:\ f(x) = 0\}\quad \text{or}\quad \{x \in R^n: f(x) \leq 0\}$$



$f: \mathbb{R}^2 \to \mathbb{R}$

eg: $B = \{x \in \mathbb{R}^2 : \|x - x_c\|^2 \leq 1\}$

$f(x) = \|x - x_c\|^2 - 1 \leq 0$

$C = \{x \in \mathbb{R}^2 : \|x - x_c\|^2 = 1\}$

- Explicit representation: $\{x(\alpha) \in R^n:\ \alpha$ satisfies certain conditions$\}$

$B = \{x_c + \alpha : \alpha \in \mathbb{R}^2, \|\alpha\| \leq 1\}$

$C = \{(\cos\alpha, \sin\alpha) ; \alpha \in [0, 2\pi)\}$

Connection with Linear algebra:

$f(x)$ linear $f(x) = Ax + b$

or quadratic $\Rightarrow f(x) = x^T Q x$

Some Simple Geometric Sets

- Line segment: Given $x_1 \neq x_2 \in R^n$: $\{x_2 + \alpha\,(x_1 - x_2): \alpha \in [0,1]\}$

- Line (explicit): $\{x_2 + \alpha\,(x_1 - x_2): \alpha \in R\}$

- Line: (implicit representation)
  - e.g. in R^2: $\{x \in R^2: a^T x = b\}$

Some Simple Geometric Sets

- Hyperplanes (Implicit): $\{x \in R^n: a^T x = b\}$

- Hyperplanes (Explicit): $\{x \in R^n: a^T x = b\} \Rightarrow \{x_0 + \sum_i \alpha_i v_i : \ \alpha_i \in R, i = 1, \dots, n-1\}$

- Halfspaces: $\{x \in R^n: a^T x \leq b\}$

Some Simple Geometric Sets

- Convex set: A set S is called convex if

$$x_1, x_2 \in S \Rightarrow \alpha x_1 + (1-\alpha) x_2 \in S$$

- **Convex combination** of $x_1, \ldots, x_k \in R^n$

$$\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k : \alpha_i \geq 0, \sum \alpha_i = 1\}$$

- Convex hull $\overline{co}(S)$: set of all convex combinations of points in S

Some Simple Geometric Sets

- **Cone**: A set S is called a cone if $x \in S \Rightarrow \lambda x \in S,\ \forall \lambda \geq 0$

- **Conic combination** of $x_1, \ldots, x_k \in R^n$

$$cone(x_1, \ldots, x_k) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \colon \alpha_i \geq 0\}$$

Some Simple Geometric Sets

- **(Convex) Polyhedron**: intersection of a finite number of half spaces

$$P = \{x : Ax \leq b\}$$

- **Polyhedral cone**: intersection of finitely many halfspaces that contain the origin:

$$P = \{x : Ax \leq 0\}$$

- **Polytope**: bounded polyhedron

Some Simple Geometric Sets

- Polyhedron (vertex representation):

$$P = \overline{co}(v_1, \dots, v_m) \oplus cone(r_1, \dots, r_q)$$

Some Simple Geometric Sets

- Euclidean balls: $B(x_c, r) = \{x \in R^n: \|x - x_c\|_2 \leq r\}$ or $B(x_c, r) = \{x_c + ru: u \in R^2, \|u\|_2 \leq 1\}$

- Ellipsoids: $E = \{x \in R^n: (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$ or $E = \{x_c + Au: u \in R^2, \|u\|_2 \leq 1\}$