MEE5114 Advanced Control for Robotics

Lecture 10: Basics of Stability Analysis

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

This lecture introduces basic concepts and results on Lyapunov stability of nonlinear systems.

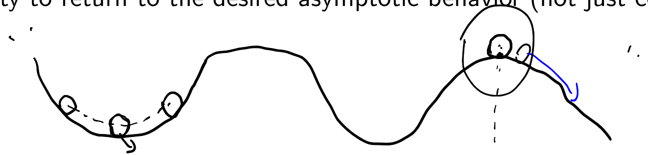
- Background
- Lyapunov Stability Definitions
- Lyapunov Stability Theorem
- Lyapunov Stability of Linear Systems
- Converse Lyapunov Function
- Extension to Discrete-Time System

Outline

What is Stability Analysis?

- system asymptotic behavior (not too much about transient)
- ability to return to the desired asymptotic behavior (not just convergence)

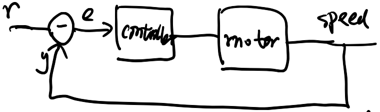
sys1: stable?



– "stable"

unstable equilibrium.

– Question X



$\dot{e}$ = error dynamics (closed-Loop)

can error converge to zero starting from nonzero I.C.

General ODE Models for Dynamical Systems

state-space form : 1^{st}-order ODE in $\mathbb{R}^n$

- ODE: $\dot{x} = f(x, u)$, with $x(0) = x_0$
  - $x \in \mathcal{X} \subseteq \mathbb{R}^n$: state
  - $u \in \mathcal{U} \subseteq \mathbb{R}^m$: control input
  - $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: (time-invariant) vector field

- System output $y = g(x, u)$

- Control law: $\mu : \mathcal{X} \to \mathcal{U}$

control law

$u = \mu(x)$

- Closed-loop dynamics under μ: $\dot{x} = f(x, \mu(x))$ $\Rightarrow$ closed-loop dynamics

$\dot{x} = f_{cL}(x)$

$\Downarrow$

autonomous sys.

- Autonomous system:

$$\dot{x} = f(x), \text{ with } x(0) = x_0 \tag{1}$$

Example: Pendulum

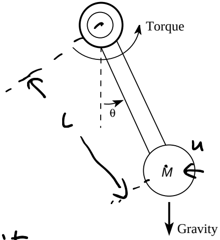
- Pendulum with driving force: $\ddot{\theta} = \underbrace{\frac{-\rho}{Ml^2}\dot{\theta}}_{\text{damping}} \cancel{+}\,- \frac{\cos\theta}{Ml}u \cancel{+}\,- \frac{g}{l}\sin\theta$

- Let $\underline{M=1,\ l=1}$, $\quad x_1 = \theta,\quad x_2 = \dot{\theta}$

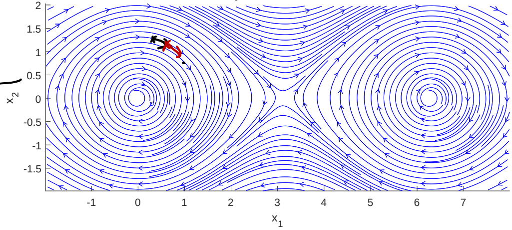
$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\rho x_2 - \cos x_1 \cdot u - g\sin x_1 \end{bmatrix} \leftarrow f(x,u)$$

$\Rightarrow$ ~~Let~~ for symplicity, $\rho = 0$, (undamped), $\boxed{u=0}$ "g" $=1$

$$\Rightarrow \boxed{\dot{x} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix} = f(x)}$$



phase portrait



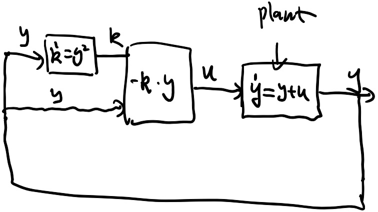
Examples: Adaptive Control

- Closed-loop dynamics under adaptive control:

$$\begin{cases} \dot{y} = y + u \\ \underbrace{u = -ky}, \dot{k} = y^2 \end{cases}$$

Linear control law

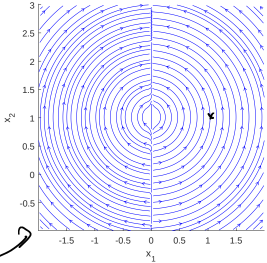
$u = -k y$



Closed-Loop dynamics: $x_1 = y$, $x_2 = k$,

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 x_1 \\ x_1^2 \end{bmatrix}}$$

$\dot{x} = f(x)$



Equilibrium Point of Dynamical Systems

$\dot{x} = f(x)$

Definition 1 (Equilibrium Point).

A state x^* is an *equilibrium point* of system (1) if once $x(t) = x^*$, it remains equal to x^* at all future time.

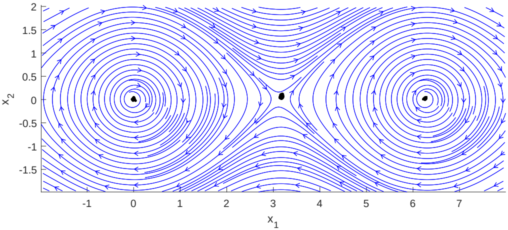
$\dot{x} = 0$ at the equilibrium

- Mathematically: $f(x^*) = 0$
- E.g undamped pendulum with no driving force:

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$

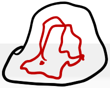
to find equilibrium:

$$\Rightarrow \left.\begin{matrix} x_2 = 0 \\ \sin x_1 = 0 \end{matrix}\right\} \Rightarrow x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \ldots\ldots$$



Invariant Set of Dynamical Systems

Definition 2 (Invariant Set). $\dot{x} = f(x)$



A set E is an *invariant set* of system (1) if every trajectory which starts from a point in E remains in E at all future time.

- Mathematically: If $x(t_0) \in E$, then $x(t) \in E$, $\forall t \geq t_0$
- E.g: closed-loop dynamics under adaptive control:

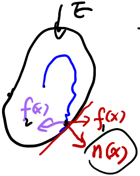
$$\begin{cases} \dot{y} = y + u \\ u = -ky, \dot{k} = y^2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} x_1 - x_1 x_2 \\ x_1^2 \end{bmatrix}$$

$f(x) = 0 \Rightarrow x_1 = 0$, x_2 arbitrary

equilibrium set

general invariant set E

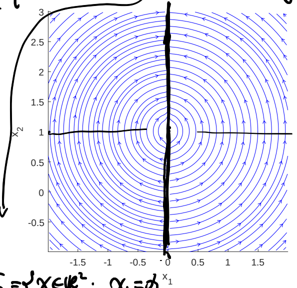


$f(x)$, $n(x)$

angle $(f(x), n(x)) \geq 90°$

$\Rightarrow$ ① $\boxed{\begin{array}{c} \langle f(x), n(x) \rangle \leq 0 \\ f^T(x)\, n(x) \leq 0 \end{array}}$

- equilibrium point: $f^T(x^*)\, n(x^*) = 0$



$E = \{x \in \mathbb{R}^2 : x_1 = 0\}$

Outline

- stability: { · about equilibrium
  - ability to stay close or return to equilibrium.

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Lyapunov Stability Definitions (1/2)

or closed-loop system $\dot{x} = f(x, u(x)) = f_{cl}(x)$

Consider a time-invariant autonomous (with no control) nonlinear system:

$$\underbrace{\dot{x} = \overset{\text{vector field}}{f(x)}}_{x \in \mathbb{R}^n} \text{ with I.C. } x(0) = x_0 \tag{2}$$

If equilibrium x^* is not at the origin define $\tilde{x} = x - x^*$

- Assumptions: (i) f Lipschitz continuous; (ii) origin is an isolated equilibrium $f(0) = 0$

  ⇒ existence & uniqueness of ODE

  $f(x^*) = 0$

  $\dot{\tilde{x}} = \dot{x} - 0 = f(x)$

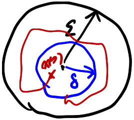
  $= f(\tilde{x} + x^*)$

- Stability Definitions: The equilibrium $x = 0$ is called
  - **stable** in the "sense of Lyapunov," if

"stay close to equilibrium"

$$\underbrace{\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0}_{\varepsilon\text{-}\delta \text{ argument.}}$$

· objective: For any $\varepsilon > 0$, ensure $\|x(t)\| \leq \varepsilon$, for all t



our choice: selecting initial state $x(0)$.

stability: objective can be ensured by choosing I.C. sufficiently small.

Lyapunov Stability Definitions (2/2)

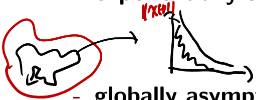
"stay close"

+ convergence

- **asymptotically stable** if it is stable and δ can be chosen so that

$$\|x(0)\| \le \delta \Rightarrow \left(x(t) \to 0 \text{ as } t \to \infty\right) \quad \text{convergence}$$

- **exponentially stable** if there exist positive constants δ, λ, c such that



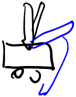
$$\|x(t)\| \le c\|x(0)\|e^{-\lambda t}, \quad \forall \|x(0)\| \le \delta$$

- **globally asymptotically/exponentially stable** if the above conditions holds for all $\delta > 0$ G.A.S. / G.E.S.

"global"

- "Region of Attraction": $R_A \triangleq \{x \in \mathbb{R}^n : \text{ whenever } x(0) = x, \text{ then } x(t) \to 0\}$

Globally asym stable $\Rightarrow$ $R_A = \mathbb{R}^n$

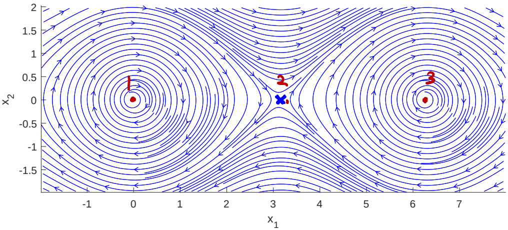


Stability Examples using 2D Phase Portrait (1/2)

- Undamped pendulum with no driving force :

$$\dot{x} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \dots$$



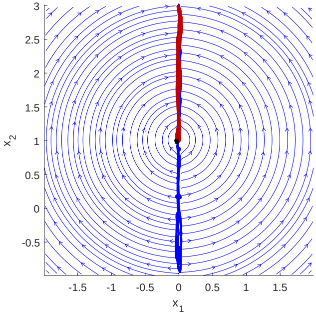
- Closed-loop dynamics under adaptive control:

$$\begin{cases} \dot{y} = y + u \\ u = -ky, \dot{k} = y^2 \end{cases}$$

x_2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 x_1 \\ x_1^2 \end{bmatrix}$$

$E = \{(x_1, x_2) : x_1 = 0\}$

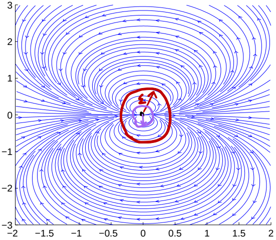


Stability Examples using 2D Phase Portrait (2/2)

Does attractiveness implies stable in Lyapunov sense? Asymp stable :

① stable ② convergence.

- Answer is NO. e.g.: $\begin{cases} \dot{x}_1 = x_1^2 - x_2^2 \\ \dot{x}_2 = 2x_1x_2 \end{cases}$



- By inspection of its vector field, we see that $x(t) \to 0$ for all $x(0) \in \mathbb{R}^2$

- However, there is no δ-ball satisfying the Lyapunov stability condition

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How to verify stability of a system? (1/2)

- Find explicit solution of the ODE $x(t)$ and check stability definitions
  - typically not possible for nonlinear systems e.g. $x(t) = e^{-t} x_0$

- Numerical simulations of ODE do not provide stability guarantees and offer limited insights

- Need to determine stability without explicitly solving the ODE

- Preferably, analysis only depends on the vector field $f(\cdot)$

How to verify stability of a system? (2/2)

- The most powerful tool is: *Lyapunov function*

- State trajectory $x(t)$ governed by complex dynamics in $\mathbb{R}^n$

  $\dot{x}(t) = f(x(t))$

- Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ maps $x(t)$ to a scalar function of time $V(x(t))$

  $\underbrace{V(x(t))}_{\text{scalar}} \Leftrightarrow \dot{V}(x(t)) = \frac{d}{dt}V(x(t)) = g(V(\cdot))$ ⟵ scalar ODE

- If the function is designed such that: $[x(t) \to \text{equilibrium}] \Leftrightarrow [V(x(t)) \to 0]$. Then we can study $V(x(t))$ as function of time t to infer stability of the state trajectory in $\mathbb{R}^n$.

Sign Definite Functions

Assume that $0 \in D \subseteq \mathbb{R}^n$

$\subseteq \mathbb{R}^n$

$\dot{x}_1$

- $g : D \to \mathbb{R}$ is called positive semidefinite (PSD) on D if $g(0) = 0$ and $g(x) \geq 0, \forall x \in D$
  - For quadratic function: $g(x) = x^T P x$: [g is PSD] $\Leftrightarrow$ [P is a PSD matrix]

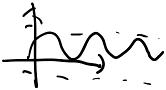
$$g(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \overset{e.g}{=} x_1^2 + x_1 x_2 + 3x_2^2$$

$\dot{x}$

- $g : D \to \mathbb{R}$ is called positive definite (PD) on D if $g(0) = 0$ and $g(x) > 0, \forall x \in D \setminus \{0\}$
  - Similarly, if $g(x) = x^T P x$ is quadratic, then [g is PD] $\Leftrightarrow$ [P is a PD matrix]

- g is negative semidefinite (NSD) if $-g$ is PSD

- $g : \mathbb{R}^n \to \mathbb{R}$ is radically unbounded if $g(x) \to \infty$ as $\|x\| \to \infty$



not radially unbounded

Lyapunov Stability Theorem

continuously differentiable

e.g. $V(x) = x_1^2$ PSD

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix} \to V(x) = 0$

[Lyapunov Theorem]: Let $D \subset \mathbb{R}^n$ be a set containing an open neighborhood of the origin. If there exists a $\mathcal{C}^1$ function $V : D \to \mathbb{R}$ such that

$\dot{V}(x(t)) = \left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t} = \nabla V(x)^T f(x)$

"observable condition"

the value of V along sys state trajectory nonincreasing

$$V \text{ is PD} \tag{3}$$

$$\dot{V}(x) \triangleq \nabla V(x)^T f(x) \text{ is NSD} \tag{4}$$

$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}$

$\triangleq \mathcal{L}_f[V]$ Lie derivative of $V(\cdot)$ with vector field f

then the origin is stable. If in addition,

$$\dot{V}(x) \triangleq \nabla V(x)^T f(x) \text{ is ND} \tag{5}$$

$\Downarrow$

Value of V along sys state traj is decreasing

then the origin is asymptotically stable.

Remarks:

- A PD $\mathcal{C}^1$ function satisfying (4) or (5) will be called a **Lyapunov function**

  (3)+ (3)+

- Under condition (5), if V is also radially unbounded
  $\Rightarrow$ globally asymptotically stable

G.A.S.

Proof of Lyapunov Stability Theorem (1/3)

Main idea: 3+4 $\Rightarrow$ stability.

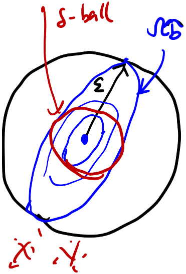
Fact: suppose V function satisfies $\boxed{③ + ④}$, then the sublevel set

$$\Omega_b(V) \triangleq \{x \in \mathbb{R}^n : V(x) \le b\} \text{ is (forward) invariant.}$$

proof Fact: if $x(0) \in \Omega_b$ for some $b \geq 0$, we have $V(x(t)) \le V(x(0)) \le b$

$\Rightarrow x(t) \in \Omega_b(V)$

- proof of stability: Given $\varepsilon > 0$, goal is to find $\delta > 0$ such that $\|x(0)\| \le \delta \Rightarrow \|x(t)\| \le \varepsilon$



1°. $\Omega_b = \{0\}$, if $b = 0$. (due to P.D. of V)

2°: As $b\uparrow$ increases, level set Ω_b grows in size until hitting B_ε. then fix $b = \hat{b}$

3°: Find B_δ inside Ω_b (because V is continuous at 0)

then $x_0 \in B_\delta \Rightarrow x_0 \in \Omega_b \Rightarrow x(t) \in \Omega_b$

$\Leftarrow \Omega_b$ is invariant

Proof of Lyapunov Stability Theorem (2/3)

Sketch of proof of Lyapunov stability theorem:

- First show stability under condition (4):
  - Define sublevel set: $\Omega_b = \{x \in \mathbb{R}^n : V(x) \leq b\}$. Condition (4) implies $V(x(t))$ nonincreasing along system trajectory $\Rightarrow$ If $x(0) \in \Omega_b$, then $x(t) \in \Omega_b$, $\forall t$.
  - Given arbitrary $\epsilon > 0$, if we can find δ, b such that $B(0,\delta) \subseteq \Omega_b \subseteq B(0,\epsilon)$. Then the Lyapunov stability conditions are satisfied. Below is to show how we can find such b and δ.
  - V is continuous $\Rightarrow$ $m = \min_{\|x\|=\epsilon} V(x)$ exists (due to Weierstrass theorem). In addition, V is PD $\Rightarrow$ $m > 0$. Therefore, if we choose $b \in (0, m)$, then $\Omega_b \subseteq B(0,\epsilon)$.
  - $V(x)$ is continuous at origin $\Rightarrow$ for any $b > 0$, there exists $\delta > 0$ such that $|V(x) - V(0)| = V(x) < b, \forall x \in B(0,\delta)$. This implies that $B(0,\delta) \subseteq \Omega_b$.

Proof of Lyapunov Stability Theorem (3/3)

- Second, show asymptotic stability under condition (5):
  - We know $V(x(t))$ decreases monotonically as $t \to \infty$ and $V(x(t)) \geq 0$, $\forall t$. Therefore, $c = \lim_{t\to\infty} V(x(t))$ exists. So it suffices to show $c = 0$. Let us use a contradiction argument.
  - Suppose $c \neq 0$. Then $c > 0$. Therefore, $x(t) \notin \Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}, \forall t$. We can choose $\beta > 0$ such that $B(0,\beta) \subseteq \Omega_c$ (due to continuitiy of V at 0).
  - Now let $a = -\max_{\beta \leq \|x\| \leq \epsilon} \dot{V}(x)$. Since V is ND, then $a > 0$
  - $V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leq V(x(0)) - a \cdot t < 0$ for sufficiently large t. $\Rightarrow$ contradiction!

Exponential Lyapunov Function

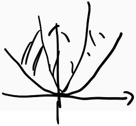
Lyapunov stability: $\exists$ C^1 func. V

$\begin{cases} V \text{ is P.D} & \text{"observable"} \\ \dot V \text{ is N.D/N.S.D.} \end{cases}$

.X: Important for applications.

Definition 3 (Exponential Lyapunov Function).

$V : D \to \mathbb{R}$ is called an Exponential Lyapunov Function (ELF) on $D \subset \mathbb{R}^n$ if $\exists k_1, k_2, k_3, \alpha > 0$ such that



$$\begin{cases} k_1\|x\|^\alpha \le V(x) \le k_2\|x\|^\alpha & \text{--- ①} \\ \underbrace{\mathcal{L}_f V(x)}_{\dot V(x(t))} \le -k_3 V(x) & \text{--- ②} \end{cases}$$

$\Rightarrow$ V is P.D. ✓ is radially unbounded

$\Rightarrow \begin{cases} \dot V \text{ is N.D.} ✓ \\ \dot V \le -k_3 V \;\cdots \end{cases}$

Theorem 1 (ELF Theorem).

If system (2) has an ELF, then it is exponentially stable.

Proof. sketch: recall: $z \in \mathbb{R}^1$, $\underbrace{\dot z = -k_3 z}_{\text{ODE}}$, $z \in \mathbb{R} \Rightarrow \boxed{z(t) = e^{-k_3 t} z(0)}$

By comparison theorem: $\boxed{\dot V \le -k_3 V \xRightarrow{\text{Fact}} V(t) \le e^{-k_3 t} V(0)}$ ——

$\Downarrow$ $V(x(t))$

$$\Rightarrow \|x(t)\|^\alpha \le \frac{1}{k_1} V(x(t)) \le \frac{1}{k_1} e^{-k_3 t} V(x(0)) \le \frac{k_2}{k_1} e^{-k_3 t} \|x(0)\|^\alpha$$

$$\Rightarrow \|x(t)\|^\alpha \le c e^{-\beta t} \|x(0)\|^\alpha$$

Stability Analysis Examples (1/2)

Example 1. $\in \mathbb{R}^2$

$$\left\{\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_1^2 - x_2^3\end{aligned}\right\} = f(x) \qquad \text{Try } V(x) = \|x\|^2 \quad \text{candidate}$$

$$= x_1^2 + x_2^2$$

- equilibrium: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad f(x) = 0 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

check Lyapunov conditions

- <1> $V(x) = x_1^2 + x_2^2$. is P.D and C^1

$$= x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

<2> $L_f V(x) = \left(\frac{\partial V}{\partial x}\right)^T f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}^T \cdot \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = 2x_1(-x_1 + x_2 + x_1 x_2) + 2x_2(x_1 - x_2 - x_1^2 - x_2^3)$

$$= 2\left(-(x_1 - x_2)^2 - x_2^4\right) \Leftarrow$$

- ~~NSD~~
- ND ~

$\Rightarrow$ system is "asym stable"

Stability Analysis Examples (2/2)

Example 2.

$$\left\{\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2\end{aligned}\right\}$$

$\Downarrow$ CDC

- Can we find a simple quadratic Lyapunov function? First try: $V(x) = x_1^2 + x_2^2$

① V is P.D.

② $L_f V(x) = -2\left((x_2-4)^2 - 8\right)$. Not ND

- In fact, the system does not have any (global) polynomial Lyapunov function. But it is GAS with a Lyapunov function $V(x) = \ln(1 + x_1^2) + x_2^2$.

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Stability of Linear Systems

$\dot{x} = Ax + Bu$

Consider autonomous linear system: $\dot{x} = f(x) = Ax$.

- Recall solution to the linear system: $x(t) = e^{At}x(0)$ ↙ analytical solution

If isolated equilibrium

- Only possible equilibrium is origin $x = 0$

  ② If A is singular, $Null(A)$ is the set of equilibrium

  $f(x)=0 \Rightarrow Ax=0$

  ① if A is nonsingular $\Rightarrow x=0$

- Fact: Origin asympt. (exponentially) stable $\Leftrightarrow$ $Re(\lambda_i) < 0$ for all eigenvalues λ_i of A

Suppose we have isolated equilibrium: $x^* = 0$

For simplicity, consider a simple case when A is diagonalizable.

$$A = TDT^{-1} \text{ where } D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Rightarrow e^{At} = Te^{Dt}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

if $Re(\lambda_i) < 0$, for all i

$\Rightarrow$ every entry of $e^{At} \to 0$

$\Rightarrow e^{At}x(0) \to 0$ exponentially.

- Discrete time system: $x(k+1) = Ax(k)$ is asymp. stable iff eig(A) inside unit circle

$e^{\lambda_i t} = e^{(-2+j)t}$

Lyapunov Function of Linear Systems

$V(x) = \|x\|^2 = x^T I x$

- Consider a quadratic Lyapunov function candidate: $V(x) = \text{``}x^T P x\text{''}$, with $P \in \mathbb{R}^{n\times n}$ $\Rightarrow P \succ 0$
  - V is PD $\Rightarrow P \succ 0$ P is a P.D. matrix
  - $\mathcal{L}_f V$ is ND $\Rightarrow$

$$L_f V(x) \triangleq \left(\frac{\partial V}{\partial x}\right)^T A x = (2Px)^T A x = 2x^T P^T A x \cdots (a)$$
$$= x^T (2P^T A) x$$

or equivalently, $\dot{V}(x(t)) = \dot{x}^T P x + x^T P \dot{x}$

$$= x^T A^T P x + x^T P A x = \underline{x^T (\underbrace{A^T P + PA}) x} \cdots (b)$$

- ~~Is $2P^TA = A^TP + PA$?~~ $\underbrace{x^T P^T A x}_{\text{scalar}} = x^T A^T P x$

$x^T A^T P x + x^T P A x = 2 x^T P^T A x \Rightarrow (a) = (b)$

$\Rightarrow$ V is L.F. if P is P.D. and $A^TP + PA$ is N.D.

[Fact: for Linear system, quadratic form of L.F. is all we need to consider.]

- ~~Inst~~ In proof of ⓐ ⇒ ⓑ, we assumed P is symmetric

so $P^T A = PA$

- e.g. $Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $g(x) = x^T Q x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$

$Q \Rightarrow \hat{Q} = \frac{1}{2} Q + \frac{1}{2} Q^T \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Fact ⇒ A is asym stable if and only if

- ① $\exists\, P \succ 0$, such that $A^T P + PA \prec 0$

→ N.D

P.D

- equivalently, ② for any $Q \succ 0$, $\exists\, P$ such that $A^T P + PA = -Q$

⇙ Lyapunov equation

Fact : ① ⇔ ② (HW)

Stability Conditions for Linear Systems

Theorem 2 (Stability Conditions for Linear System).

For an autonomous Linear system $\dot{x} = Ax$. The following statements are equivalent.

- *System is (globally) asymptotically stable*
- *System is (globally) exponentially stable*

for linear system

- $Re(\lambda_i) < 0$ *for all eigenvalues* λ_i *of* A

lie on $\underbrace{\text{open left half}}_{\text{OLHP}}$ complex plane.

- *System has a quadratic Lyapunov function*

$V(x) = x^T P x$

- *"For any symmetric $Q \succ 0$, there exists a symmetric $P \succ 0$ that solves the following Lyapunov equation":*

$$\boxed{PA + A^T P = -Q}$$

$\leftarrow$ $Q \succ 0$ is given

$\leftarrow$ P is the variable to be solved.

and $V(x) = x^T P x$ is a Lyapunov function of the system.

Outline

- Background
- Lyapunov Stability Definitions
- Lyapunov Stability Theorem
- Lyapunov Stability of Linear Systems
- Converse Lyapunov Function
- Extension to Discrete-Time System

- For nonlinear sys, $\exists V \Rightarrow$ stability (sufficient condition)

$\exists V$ " $\Leftarrow$ "

When There is a Lyapunov Function?

- Converse Lyapunov Theorem for Asymptotic Stability

$$\begin{cases} \text{origin asymptotically stable;} \\ f \text{ is locally Lipschitz on } D \\ \text{with region of attraction } R_A \end{cases} \Rightarrow \exists V \text{ s.t.} \begin{cases} \text{V is continuuos and PD on } R_A \\ \mathcal{L}_f V \text{ is ND on } R_A \\ V(x) \to \infty \text{ as } x \to \partial R_A \end{cases}$$

converse result that is not constructive

- Converse Lyapunov Theorem for Exponential Stability

$$\begin{cases} \text{origin exponentially stable on } D; \\ f \text{ is } \mathcal{C}^1 \end{cases} \Rightarrow \exists \text{ an ELF } V \text{ on } D$$

- Proofs are involved especially for the converse theorem for asymptotic stability

- **IMPORTANT**: proofs of "converse theorems" often assume the knowledge of system solution and hence are not constructive.

Outline

- Background
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What about Discrete Time Systems?

- So far, all our definitions, results, examples are given using continuous time dynamical system models.

- All of them have discrete-time counterparts. The ideas and conclusions are the "same" (in spirit)

- For example, given autonomous discrete-time system: $x(k+1) = f(x(k))$ with $f(0) = 0$ (origin is an isolated equilibrium).
  - Rate of change of a function $V(x)$ along system trajectory can be defined as:

C.T.: $L_f V(x) \triangleq (\frac{\partial V}{\partial x})^T f(x)$

$$\Delta_f V(x) \triangleq V(f(x)) - V(x) \Leftarrow V(x(k+1)) - V(x(k))$$

  - Asymptotically stable requires:

"observable" all the bad behavior of x shows up in V

$$\text{"}V \text{ is PD"} \quad \text{and} \quad \Delta_f V \text{ is ND}$$

$\Delta_f V(x) < 0$ for all $x \in \mathbb{R}^n / \{0\}$

  - Exponentially stable requires:

$L_f V$

$$k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \quad \text{and} \quad \Delta_f V(x) \leq -k_3 V(x)$$

  - . . . .

Concluding Remarks

- We have learned different notions of internal stability, e.g. stability in Lyapunov sense, asymptotic stability, globally asymptotic stability (G.A.S), exponential stability, globally exponential stability (G.E.S)

- Sufficient condition to ensure stability is often the existence of a properly defined Lyapunov function

- Key requirements for a Lyapunov function:
  - positive definite and is zero at the system equilibrium
  - decrease along system trajectory

- For linear system: G.A.S $\Leftrightarrow$ G.E.S $\Leftrightarrow$ Existence of a quadratic Lyapunov function

- The definitions and results in this lecture have sometimes been stated in simplified forms to facilitate presentation. More general versions can be found in standard textbooks on nonlinear systems

- ~~**Next Lecture**~~: Semidefinite Programming and computational stability analysis

More Discussions

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More Discussions

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