MEE5114 Advanced Control for Robotics

Lecture 11: "Basics" of Optimization

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Outline

- Motivation
- Some Linear Algebra
- Sets and Functions
- Short Introduction to Optimization
- Linear Program
- Quadratic Program

Motivation

- Optimization is arguably the most important tool for modern engineering
- Robotics
  - Differential Inverse Kinematics
  - Dynamics : ABA ⟺ LQR
  - Motion planning
  - Whole-body control: formulated as a quadratic program
  - SLAM:
  - Perception
- Machine Learning
  - Linear regression
  - Support vector machine:
  - Deep learning minimize "loss" func.
- other domains
  - Check system stability: SDP
  - Compressive sensing
  - Fourier transform: least square problem
- Roughly speaking, most engineering problems (finding a better design, ensure certain properties of the solution, develop an algorithm), can be formulated as optimization/optimal control problems.

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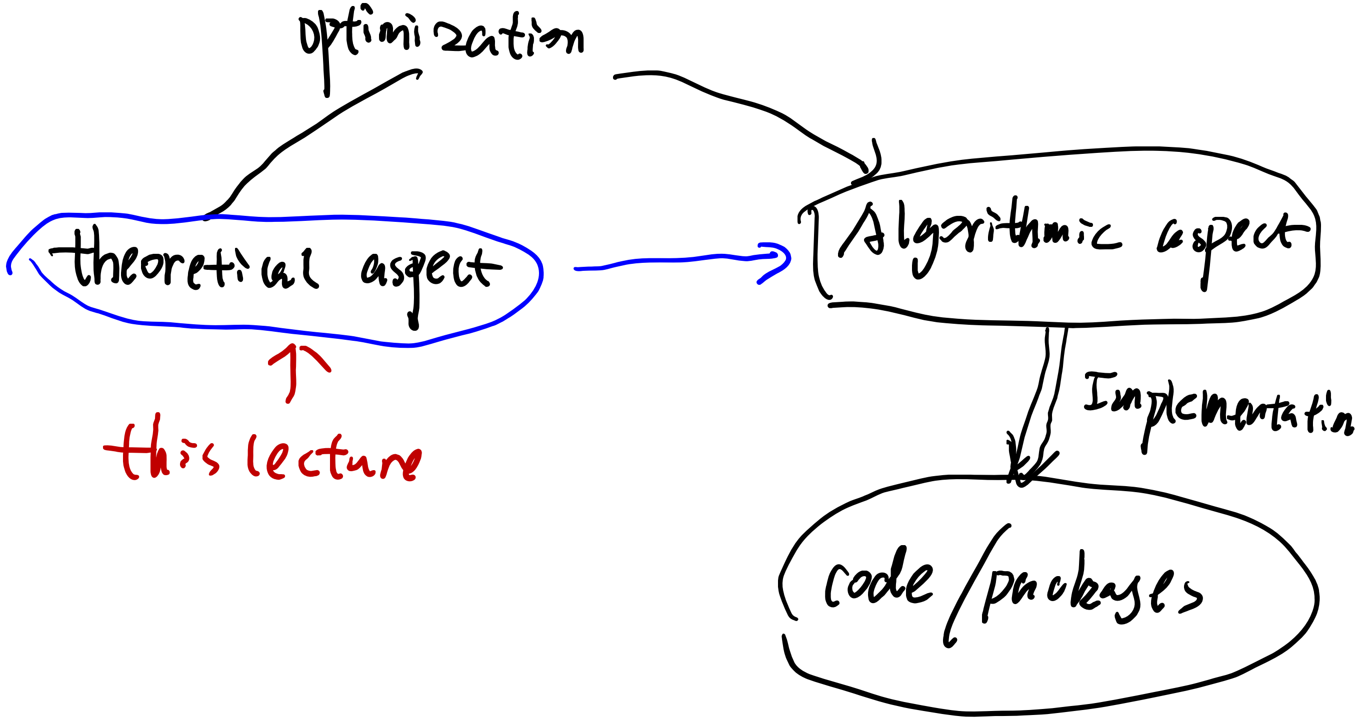

Our goal: { Basic knowledge / key concepts of opt. theory
- Formulate / reformulate opt. prob
- Educated users of tools / packages

Real Symmetric Matrices

- $\mathcal{S}^n$: set of real symmetric matrices

- All eigenvalues are real

- There exists a full set of orthogonal eigenvectors

- *Spectral decomposition*: If $A \in \mathcal{S}^n$, then $A = Q\Lambda Q^T$, where Λ diagonal and Q is unitary.

Positive Semidefinite Matrices (1/3)

- $A \in \mathcal{S}^n$ is called *positive semidefinite (p.s.d.)*, denoted by $A \succeq 0$, if $x^T A x \geq 0$, $\forall x \in \mathbb{R}^n$
- $A \in \mathcal{S}^n$ is called *positive definite (p.d.)*, denoted by $A \succ 0$, if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$
- $\mathcal{S}^n_+$: set of all p.s.d. (symmetric) matrices
- $\mathcal{S}^n_{++}$: set of all p.d. (symmetric) matrices
- p.s.d. or p.d. matrices can also be defined for non-symmetric matrices.
  e.g. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

  $x^T \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x = x_1^2 + x_2^2$

- We assume p.s.d. and p.d. are symmetric (unless otherwise noted)
- Notation: $A \succeq B$ (resp. $A \succ B$) means $A - B \in \mathcal{S}^n_+$ (resp. $A - B \in \mathcal{S}^n_{++}$)

defined a "partial order" on S^n — A - B p.s.d.

it's possible to have $A \not\succeq B$ $A \not\preceq B$

Positive Semidefinite Matrices (2/3)

- Other equivalent definitions for symmetric p.s.d. matrices:
  - All $2^n - 1$ principal minors of A are nonnegative
  - All eigs of A are nonnegative
  - There exists a factorization $A = B^T B$
- Other equivalent definitions for p.d. matrices:
  - All n leading principal minors of A are positive
  - All eigs of A are strictly positive
  - There exists a factorization $A = B^T B$ with B square and nonsingular.

$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}$

- If $A \succ 0$, $\Rightarrow A = Q\Lambda Q^T = Q\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}Q^T = B^T B$

eigenvalues > 0 $\quad B = \Lambda^{\frac{1}{2}} Q^T$

Positive Semidefinite Matrices (3/3)

- Useful facts: P.D.
  - If T nonsingular, $A \succ 0 \Leftrightarrow T^T AT \succ 0$; and $A \succeq 0 \Leftrightarrow T^T AT \succeq 0$

    doesn't need to unitary

    Recall: TAT^{-1} : similarity transformation; T^TAT : congruent transformation

    S_+^n, S_{++}^n are invariant under congruent transformation
  - Inner product on $\mathbb{R}^{m\times n}$: $< A, B > \triangleq tr(A^T B) \triangleq A \bullet B$.

    $\forall$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$ $tr(A^TB) \overset{?}{=} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$

    Angle ~~becomes~~ between A, B $\cos\theta = \frac{\langle A,B\rangle}{\sqrt{\langle A,A\rangle \langle B,B\rangle}}$

    $A \perp B \Rightarrow tr(A^TB) = 0$

    $tr(A^TB) > 0 \Rightarrow$ acute
  - For $A, B \in S_+^n$, $tr(AB) \geq 0$

    A, B square, symmetric, p.s.D.

    $\langle A,B\rangle = tr(A^TB) = tr(AB) \xrightarrow{\text{Fact}} tr(AB) \geq 0$ → HW

Positive Semidefinite Matrices (4/1)

- For any symmetric $A \in \mathcal{S}^n$,

$$\lambda_{\min}(A) \geq \mu \Leftrightarrow A \succeq \mu I \quad \text{and} \quad \lambda_{\max}(A) \leq \beta \Leftrightarrow A \preceq \beta I$$

$A - \mu I \succeq 0$

proof: $A \in S^n$

$\Rightarrow A = Q \Lambda Q^T$, for unitary Q.

$$A - \mu I = Q(\Lambda - \mu I)Q^T$$

$$A - \mu I \succeq 0 \Leftrightarrow \Lambda - \mu I \succeq 0$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{bmatrix} - \begin{bmatrix} \mu & & \\ & \mu & \\ & & \ddots & \\ & & & \mu \end{bmatrix} \succeq 0$$

$$\Leftrightarrow \lambda_{\min}(A) \geq \mu$$

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Affine Sets and Functions (1/3)

constant $f(x) \equiv c$

- Linear mapping: $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha x$ ($\alpha f(x)$), for any x, y in some vector space, and $\alpha \in \mathbb{R}$

Examples

  - $f(x) = Ax$, $x \in \mathbb{R}^3$, $A \in SO(3)$ ← check definitions

    $f(x+y) = A(x+y) = Ax + Ay = f(x) + f(y)$.

  - $f[x] = \int x(\tau)d\tau$, for all integrable function $x(\cdot)$

    integrable function of t.

    $f[x+y] = \int (x(\tau) + y(\tau))d\tau = \int x(\tau)d\tau + \int y(\tau)d\tau = f[x] + f[y]$

  - $E(x)$ expection of a random variable/vector x

    $E(X) = \int x f(x) dx$

  - $f(x) = \text{tr}(x)$, $x \in R^{n \times n}$

    $tr(A+B) = tr(A) + tr(B)$ $\quad$ $tr(\alpha A) = \alpha tr(A)$

    $\min_x tr(X)$ ⇐ L.P.
    s.t. ...

Affine Sets and Functions (2/3)

- Affine mapping: $f(x)$ is an affine mapping of x if $g(x) \triangleq f(x) - f(x_0)$ is a linear mapping for some fixed x_0

- Finite-dimension representation of affine function: $f(x) = Ax + b$

$g(x) = f(x) - f(0) = Ax + b - b = Ax$

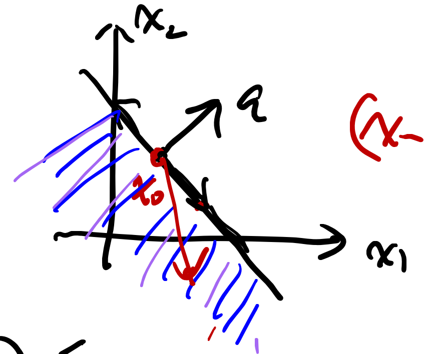
- Homogeneous representation in $\mathbb{R}^n$:

$$f(x) = Ax + b \quad \Leftrightarrow \quad \tilde{f}(\tilde{x}) = \tilde{A}\tilde{x},$$

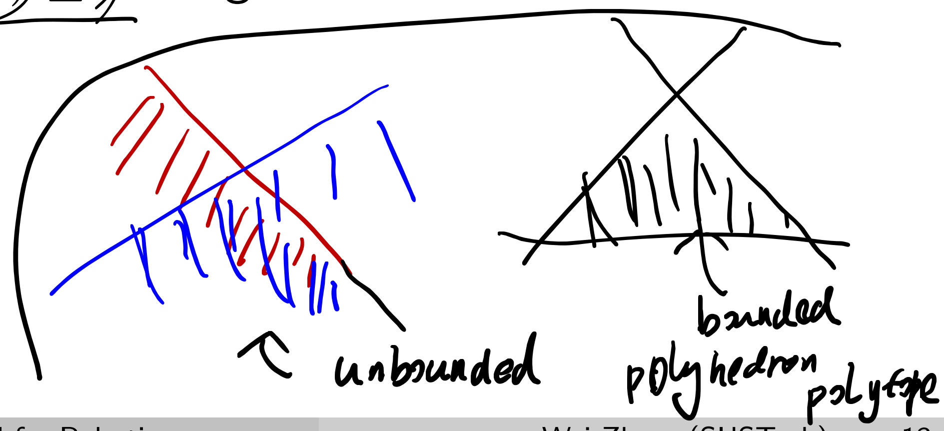
$$\text{with } \tilde{A} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}, \tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

- Linear and affine are often used interchangeably

Affine Sets and Functions (3/1)

- Linear/affine sets: $\{x : f(x) \leq 0\}$ for affine mapping f
  - Line/hyperplane: $a^T x = b$
  

  - Half space: $a^T x \leq b$
  - Polyhedron: $Hx \leq h$
  - For matrix variable $X \in \mathbb{R}^{n\times n}$, $\text{tr}(AX) \leq 0$ for given constant matrix $A \in \mathbb{R}^{n\times n}$ is a halfspace in $\mathbb{R}^{n\times n}$



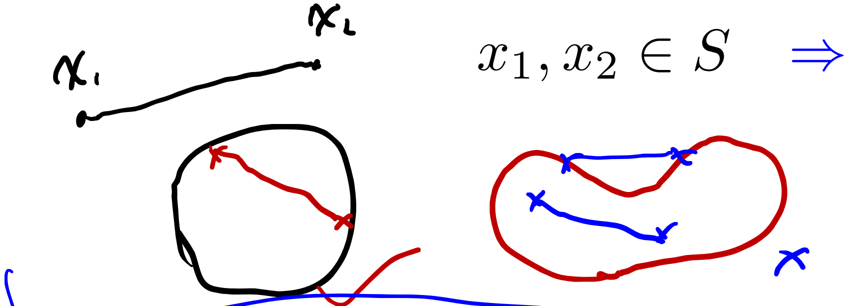
Quadratic Sets and Functions

- Quadratic functions in $\mathbb{R}^n$: $f(x) = x^T A x + b^T x + c$
- Quadratic functions (homogeneous form): $f(x) = x^T A x$
  - $A \in \mathcal{S}_+ \Leftrightarrow f(x) \geq 0, \forall x \in \mathbb{R}^n$
- Quadratic sets: $\{x :\in \mathbb{R}^n : f(x) \leq 0\}$ for some quadratic function f
  - e.g.: Ball:
  - e.g.: Ellipsoid:

Convex Set

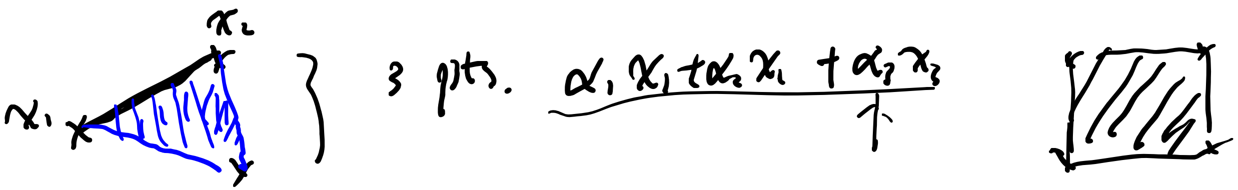
- *Convex Set*: A set S is convex if any line segment stays in the set.

$$x_1, x_2 \in S \Rightarrow \alpha x_1 + (1-\alpha)x_2 \in S, \forall \alpha \in [0,1]$$

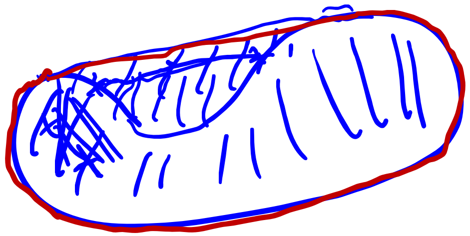


- *Convex combination* of $x_1, \ldots, x_k$:

$$\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k : \alpha_i \geq 0, \text{ and } \sum_i \alpha_i = 1 \right)$$

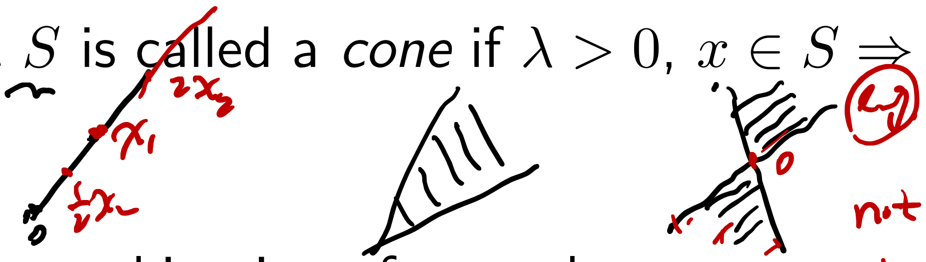


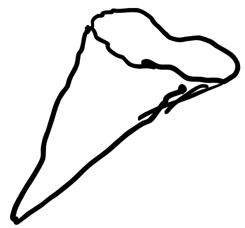
- *Convex hull*: $\overline{co}\{S\}$ set of all convex combinations of points in S



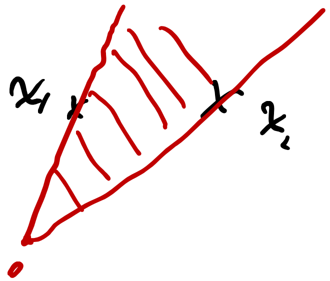
Cone

- A set S is called a *cone* if $\lambda > 0$, $x \in S \Rightarrow \lambda x \in S$.





- Conic combination of x_1 and x_2:
  $x = \alpha_1 x_1 + \alpha_2 x_2$ with $\alpha_1, \alpha_2 \geq 0$



- *Convex cone:*
  1. a cone that is convex
  2. equivalently, a set that contains all the conic combinations of points in the set

Positive Semidefinite Cone

Define $S_n^+ = S_+^n$

- The set of positive semidefinite matrices (i.e. $\mathcal{S}_+^n$) is a convex cone and is referred to as the *positive semidefinite (PSD) cone*

S_+^n: set of P.S.D. :

pick $A \in S_+^n \Rightarrow \lambda A \succeq 0 \Rightarrow \lambda A \in S_+^n$ ($\lambda \geq 0$)

S_+^n is cone.

By definition:

Pick arbitrary $A, B \in S_+^n$, $\alpha A + (1-\alpha) B \in S_+^n$? $\alpha \in [0,1]$

By definition of P.S.D., $x^T(\alpha A + (1-\alpha)B)x = \alpha (x^T A x) + (1-\alpha)(x^T B x) \geq 0$

$\Downarrow$

\# Q.E.D.

- Recall that if $A, B \in \mathcal{S}_+^n$, then $tr(AB) \geq 0$. This indicates that the cone $\mathcal{S}_+^n$ is acute.

$x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$

$\alpha_1 A + \alpha_2 B$, $\alpha_1 + \alpha_2 = 1$

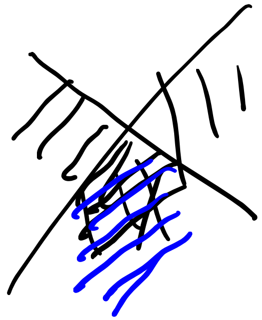
- $\alpha_1 x_1 + \alpha_2 x_2$, linear combination

- $\alpha_1 x_1 + \alpha_2 x_2$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, conic combination

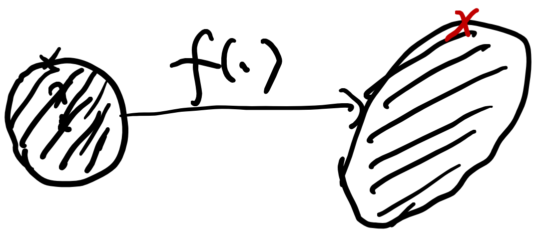
- $\alpha_1 x_1 + \alpha_2 x_2$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$ convex combination

Operations that Preserve Convexity (1/1)

- Intersection of possibly infinite number of convex sets:
  - e.g.: polyhedron:



  - e.g.: PSD cone:
- Affine mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ (i.e. $f(x) = Ax + b$)
  - $f(X) = \{f(x) : x \in X\}$ is convex whenever $X \subset \mathbb{R}^n$ is convex
    e.g.: Ellipsoid: $E_1 = \{x \in \mathbb{R}^n : (x - x_c)^T P(x - x_c) \leq 1\}$ or equivalently
    $E_2 = \{x_c + Au : \|u\|_2 \leq 1\}$
  - $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\}$ is convex whenever $Y \subseteq \mathbb{R}^m$ is convex
    e.g.: $\{Ax \leq b\} = f^{-1}(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+$ is nonnegative orthant



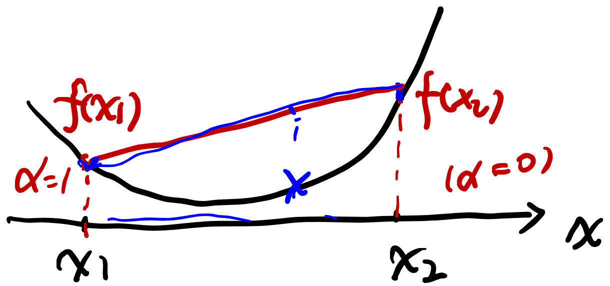
Convex Function

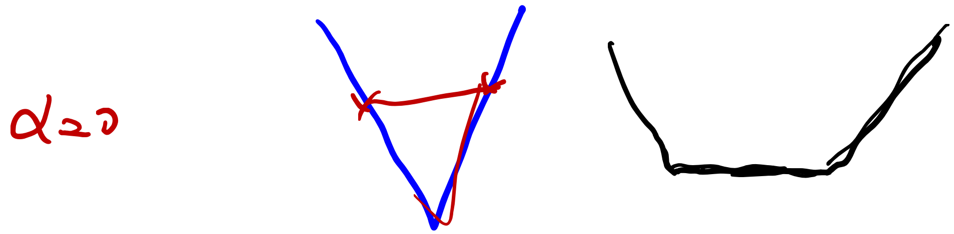
Consider a finite dimensional vector space $\mathcal{X}$. Let $\mathcal{D} \subset \mathcal{X}$ be convex.

Definition 1 (Convex Function).

A function $f : \mathcal{D} \to \mathbb{R}$ is called convex if

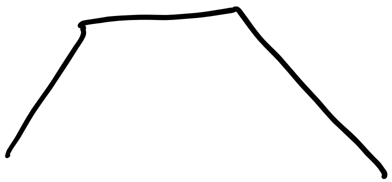
$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2), \forall x_1, x_2 \in \mathcal{D}, \forall \alpha \in [0,1]$$





- $f : \mathcal{D} \to \mathbb{R}$ is called strictly convex if
$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2), \forall x_1 \neq x_2 \in \mathcal{D}, \forall \alpha \in [0,1]$

- $f : \mathcal{D} \to \mathbb{R}$ is called concave if $-f$ is convex



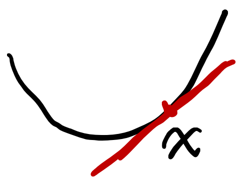
How to Check a Function is Convex?

- Directly use definition

- First-order condition: If f is differentiable over an open set that contains $\mathcal{D}$, then f is convex over $\mathcal{D}$ iff

stay above Taylor around x

$$f(z) \geq f(x) + \nabla f(x)^T (z - x), \forall x, z \in \mathcal{D}$$



- Second-order condition: Suppose f is twicely differentiable over an open set that contains $\mathcal{D}$, then f is convex over $\mathcal{D}$ iff

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \mathcal{D}$$

concave $\nabla^2 f(x) \preceq 0, \forall x \in D.$

- Many other conditions, tricks,...

Examples of Convex Functions

- In general, affine functions are both convex and concave
  - e.g.: $f(x) = a^T x + b$, for $x \in \mathbb{R}^n$ $\quad \nabla^2 f(x) = 0$
  - e.g.: $f(X) = tr(A^T X) + c = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + c$, for $X \in \mathbb{R}^{m\times n}$ (m×n matrix)

    $f: \mathbb{R}^{m\times n} \to$ scalar / affine func of X (matrix)
- Quadratic functions: $f(x) = x^T Q x + b^T x + c$ is convex iff $Q \succeq 0$

  using 2nd-order condition

  $\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ \vdots & & \end{bmatrix} = Q$ (n×n matrix), $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$
- All norms are convex
  - e.g. in $\mathbb{R}^n$: $f(x) = \|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$; $f(x) = \|x\|_\infty = \max_k |x_k|$
  - e.g. in $\mathbb{R}^{m\times n}$: $f(X) = \|X\|_2 = \sigma_{\max}(X)$

  $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

  $\|x\|_\infty = \max_k |x_k|$

  $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$

- Affine mapping of convex func is still convex.

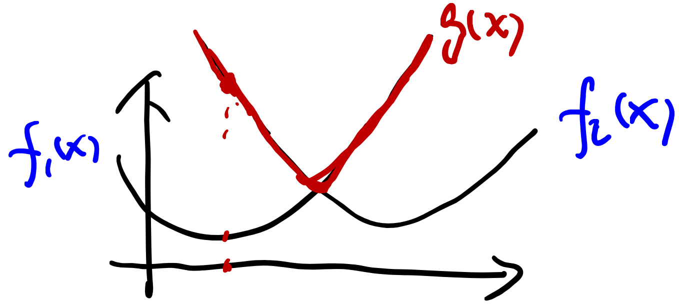
  e.g. suppose $f(x)$ convex $\Rightarrow$ $g(x) \triangleq a f(x) + b$ is also convex

  HW#

- Pointwise maximum of convex func is convex

  suppose $f_1(x)$, $f_2(x)$ are convex $\Rightarrow$ $g(x) \triangleq \max\{f_1(x), f_2(x)\}$

  then g is convex



suppose $f(x;\theta)$ is convex for each $\theta \in [1,2]$

then $g(x) \triangleq \max_{\theta \in [1,2]} f(x;\theta)$ convex

e.g. $f(x;\theta) = \theta x + b \Rightarrow g(x) = \max_{\theta \in [1,2]} \theta x + b$

- Pointwise minimum of concave func is concave.

  $g(x) = \min_{\theta \in [1,2]} \theta x + b$ is concave

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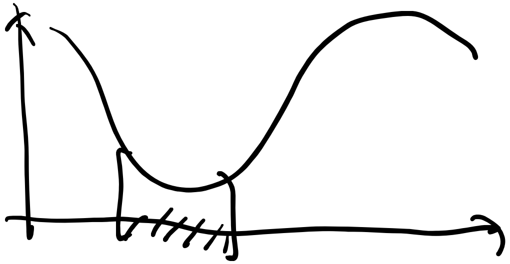
Nonlinear Optimization Problems

Nonlinear Optimization:



$$\begin{array}{ll} \text{minimize:} & f_0(x) \\ \text{subject to:} & f_i(x) \le 0, i = 1, \ldots m \\ & h_i(x) = 0, i = 1, \ldots, q \end{array}$$

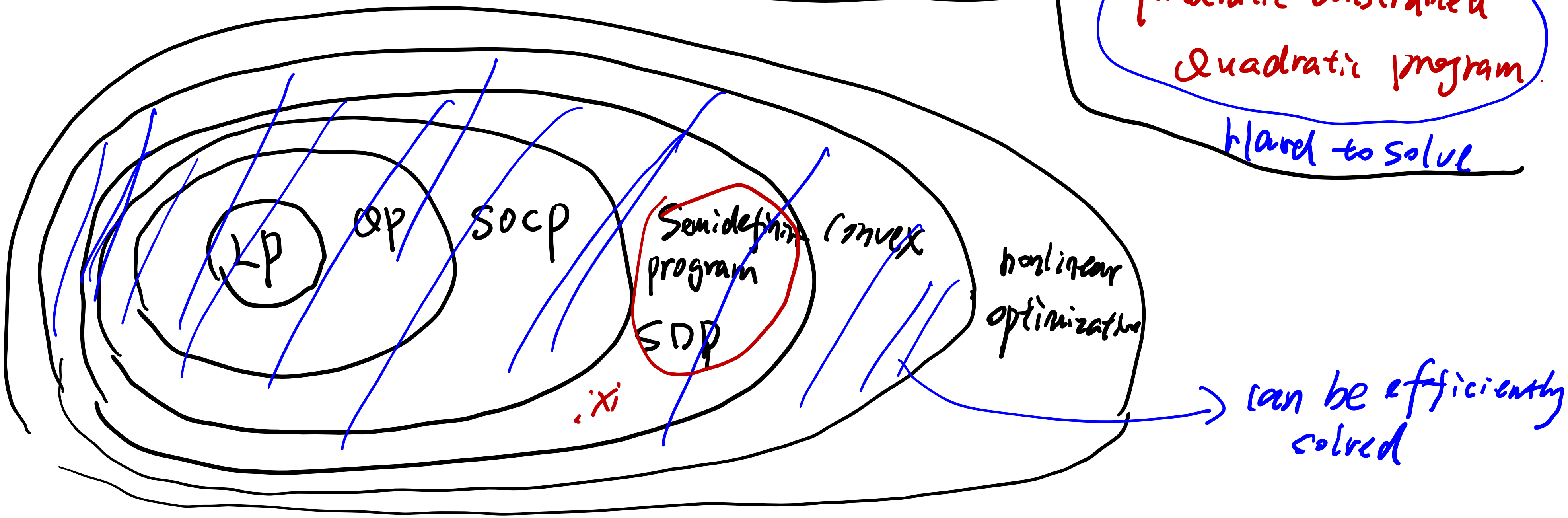
- decision variable $x \in \mathbb{R}^n$, domain $\mathcal{D}$, referred to as *primal problem*
- optimal value p^*
- is called a convex optimization problem if $f_0, \ldots, f_m$ are convex and $h_1, \ldots, h_q$ are affine
- typically convex optimization can be solved efficiently

Nonlinear Optimization Problems

- Categories:

objective func	constraint set/func	
Linear/affine	Linear/affine	→ Linear Program (LP)
Quadratic (convex)	Linear/affine	→ Quadratic Program (QP)
Quadratic (convex)	quadratic	→ QCQP quadratic constrained Quadratic program

Hard to solve



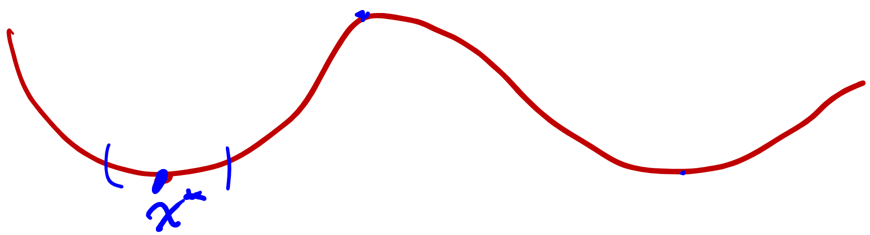
- How to find optimal sln?

  - optimality condition: for unconstrained problems (general)

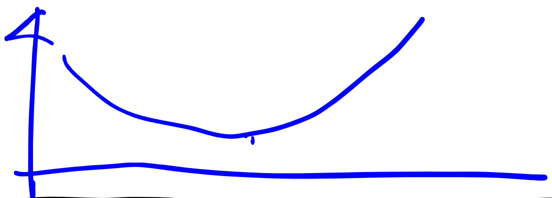
  - 1^{st}-order optimality condition: x^* is local minimizer

    then $\nabla f(x^*) = 0$ ($\in \mathbb{R}^n$) ... ① $\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \end{bmatrix}$ → suppose $\frac{\partial f}{\partial x_1} > 0$, then $x = \begin{bmatrix} x_1^* - \varepsilon \\ \vdots \\ \vdots \end{bmatrix}$

    Taylor expansion: $f(x) \approx f(x^*) + (\nabla f(x^*))^T (x - x^*) + \text{H.O.T.}$



  - For convex problem, condition ① guarantees x^* is global minimizer



Question: what about constrained optimization?

Lagrangian

Associated **Lagrangian**: $L : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{q} \nu_i h_i(x),$$

objective func

require $\lambda_i \geq 0, \forall i$

- weighted sum of objective and constraints functions

- λ_i: Lagrangian multiplier associated with $f_i(x) \leq 0$

- ν_i: Lagrangian multiplier associated with $h_i(x) = 0$

Lagrange Dual Problems (1/2)

Lagrange dual function: $g : \mathbb{R}^m \times \mathbb{R}^q :\to \mathbb{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

$$= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{q} \nu_i h_i(x) \right)$$

- g is concave, can be $-\infty$ for some λ, ν
- **Lower bound property**: If $\lambda \succeq 0$ (elementwise), then $g(\lambda, \nu) \leq p^*$

Lagrange Dual Problems (2/1)

Lagrange Dual Problem:

$$\begin{aligned} \text{maximize}: \quad & g(\lambda, \nu) \\ \text{subject to}: \quad & \lambda \succeq 0 \end{aligned}$$

- Find the best lower bound on p^* using the Lagrange dual function
- a convex optimization problem even when the primal is nonconvex
- optimal value denoted d^*
- (λ, ν) is called **dual feasible** if $\lambda \succeq 0$ and $(\lambda, \nu) \in \textbf{dom}(g)$
- Often simplified by making the implicit constraint $(\lambda, \nu) \in \textbf{dom}(g)$ explicit

Duality Theorems

- **Weak Duality:** $d^* \leq p^*$
  - always hold (for convex and nonconvex problems)
  - can be used to find nontrivial lower bounds for difficult problems
- **Strong Duality:** $d^* = p^*$
  - not true in general, but typically holds for convex problems
  - conditions that guarantee strong duality in convex problems are called *constraint qualifications*
  - Slater's constraint qualification: Primal is strictly feasible

i.e. $\exists \tilde{x}$ such $f_i(\tilde{x}) < 0$, $h_i(\tilde{x}) = 0$

General Optimality Conditions (1/3)

For general optimization problem:

$$
\begin{aligned}
\text{minimize:} \quad & f_0(x) \\
\text{subject to:} \quad & f_i(x) \le 0, i = 1, \dots m \\
& h_i(x) = 0, i = 1, \dots, q
\end{aligned}
$$

General optimality condition:

strong duality and (x^*, λ^*, ν^*) **is primal-dual optimal** $\Leftrightarrow$

- $x^* = \arg\min_x L(x, \lambda^*, \nu^*)$ (Lagrange optimality)
- $\lambda_i^* f_i(x^*) = 0$ for all i (Complementarity)
- $f_i(x^*) \le 0 \; h_j(x^*) = 0$, for all i, j (primal feasibility)
- $\lambda_i^* \ge 0$ for all i (dual feasibility)

General Optimality Conditions (2/3)

Proof of Necessity

- Assume x^* and (λ^*, ν^*) are primal-dual optimal slns with zero duality gap,

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, \nu^*) \\
&= \min_{x \in \mathcal{D}} \; f_0(x) + \sum_i \lambda_i^* f_i(x) + \sum_j \nu_j^* h_j(x) \\
&\leq f_0(x^*) + \sum_i \lambda_i^* f_i(x^*) + \sum_j \nu_j^* h_j(x^*) \\
&\leq f_0(x^*)
\end{aligned}
$$

- Therefore, all inequalities are actually equalities

- Replacing the first inequality with equality $\Rightarrow x^* = \text{argmin}_x L(x, \lambda^*, \nu^*)$

- Replacing the second inequality with equality $\Rightarrow$ complementarity condition

General Optimality Conditions (3/1)

Proof of Sufficiency

- Assume (x^*, λ^*, ν^*) satisfies the optimality conditions:

$$g(\lambda^*, \nu^*) = f(x^*) + \sum_i \lambda_i^* f_i(x^*) + \sum_j \nu_j^* h_j(x^*)$$
$$= f(x^*)$$

- The first equality is by Lagrange optimality, and the 2nd equality is due to complementarity

- Therefore, the duality gap is zero, and (x^*, λ^*, ν^*) is the primal dual optimal solution

KKT Conditions

For **convex** optimization problem:

$$
\begin{aligned}
\text{minimize:} \quad & f_0(x) \\
\text{subject to:} \quad & f_i(x) \leq 0, i = 1, \ldots m \\
& h_i(x) = 0, i = 1, \ldots, q
\end{aligned}
$$

Suppose duality gap is zero, then (x^*, λ^*, ν^*) is primal-dual optimal if and only if it satisfies the Karush-Kuhn-Tucker (KKT) conditions

- $\frac{\partial L}{\partial x}(x, \lambda^*, \nu^*) = 0$ due to primal convexity (Stationarity)
- $\lambda_i^* f_i(x^*) = 0$ for all i (Complementarity)
- $f_i(x^*) \leq 0 \; h_j(x^*) = 0$, for all i, j (primal feasibility)
- $\lambda_i^* \geq 0$ for all i (dual feasibility)

Outline

- Motivation
- Some Linear Algebra
- Sets and Functions
- Short Introduction to Optimization
- Linear Program
- Quadratic Program

Linear Program: Primal and Dual Formulations

- **Primal Formulation:**

$$\begin{aligned} \underset{x}{\text{minimize:}} \quad & c^T x \\ \text{subject to:} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$x_1 \geq 0, \ldots, x_n \geq 0$

n variables
q – equality constraint
n – inequalities $f_i(x) \leq 0$

$(-x) \leq 0$

Lagrangian func: $L(x, \lambda, \nu) = c^T x + \lambda^T(-x) + \nu^T(Ax - b)$

$$\Rightarrow g(\lambda, \nu) \triangleq \inf_{x \in \mathbb{R}^n} \left\{ (c^T - \lambda^T + \nu^T A)x - \nu^T b \right\}$$

$\min_{x \in \mathbb{R}} 2x + 3$

$$= \begin{cases} -\infty & \text{if } c^T - \lambda^T + \nu^T A \neq 0 \\ -b^T \nu & \text{if } c^T - \lambda^T + \nu^T A = 0 \end{cases}$$

$\lambda, \nu \in \mathrm{dom}(g)$

- **Its Dual:**

$$\begin{aligned} \text{maximize:} \quad & -b^T \nu \\ \text{subject to:} \quad & A^T \nu + c \geq 0 \end{aligned}$$

q – variables
n – inequality constraint

$\max_{\lambda, \nu} g(\lambda, \nu)$

subject to: $\lambda \geq 0$

$c^T - \lambda^T + \nu^T A = 0$

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Unconstrained Quadratic Program: Least Squares

- minimize: $J(x) = \frac{1}{2}x^T Q x + q^T x + q_0$
- Problem is convex iff $Q \succeq 0$
- When J is convex, it can be written as: $J(x) = \|Q^{\frac{1}{2}}x - y\|^2 + c$
- KKT condition:
- Optimal solution:

Equality Constrained Quadratic Program

- Standard form: $$\begin{cases} \min_x & J(x) = x^T Q x + q^T x + q_0 \\ \text{subject to:} & Hx = h \end{cases}$$
- The problem is convex if $Q \succeq 0$
- KKT Condition:
- Optimal Solution:

General Quadratic Program

- Standard form: $$\begin{cases} \text{minimize:} & J(x) = x^T Q x + q^T x + q_0 \\ \text{subject to:} & Ax \leq b \end{cases}$$

- Dual problem:

More Discussions

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More Discussions

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