MEE5114 Advanced Control for Robotics

Lecture 12: Semidefinite Programming for Stability Analysis

SDP

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

$f_i(x) \leq 0$, $\{x \in \mathbb{R}^n : f_i(x) \leq 0\}$

- Linear Matrix Inequalities
- Semidefinite Programming Problems
- S-Procedure
- Some Examples
- Conclusion

Outline

- Linear Matrix Inequalities
- Semidefinite Programming Problems
- S-Procedure
- Some Examples
- Conclusion

Linear Matrix Inequalities (1/4)

known

- *Standard form*: Given symmetric matrices $\overbrace{F_0, \ldots, F_m}^{\text{known}} \in \mathcal{S}^n$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x \in \mathbb{R}^m$

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0$$

is called a *Linear Matrix Inequality* in $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$

$x \in \mathbb{R}^2$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $F(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0$,

$= \begin{bmatrix} 1 + x_1 + 2x_2 & 1 \\ x_1 & 1 - x_2 \end{bmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$

$F : \mathbb{R}^2 \to S^2$

- The function $F(x)$ is affine in x

$F(x) = F_0 + G(x)$ → linear term

$G(\alpha x) = \alpha G(x)$

- The constraint set $\{x \in \mathbb{R}^n : F(x) \succeq 0\}$ is nonlinear but convex

$E \triangleq$

show E is convex : pick $x, y \in E$, $\underbrace{\alpha x + (1-\alpha) y \in E}_{\text{we need to show}}$, $\forall \alpha \in [0, 1]$

$\left.\begin{matrix} x \in E \\ y \in E \end{matrix}\right\} \Rightarrow \begin{matrix} F(x) \succeq 0 \\ F(y) \succeq 0 \end{matrix}$, we need to show $F(\alpha x + (1-\alpha) y) \succeq 0$?

check: $F(\alpha x + (1-\alpha) y) = F_0 + \alpha G(x) + (1-\alpha) G(y)$

$$= \alpha\left(F_0 + G(x)\right) + (1-\alpha)\left(F_0 + G(y)\right)$$

$$= \alpha \underbrace{F(x)}_{\in S_+^n} + (1-\alpha) \underbrace{F(y)}_{\in S_+^n} \succeq 0$$

due to S_+^n is convex

P.S.D cone

Linear Matrix Inequalities (2/4)

$= (\alpha + 1 - \alpha)F_0 + \cdots = \alpha(F_0 + G(x)) + (1-\alpha)(F_0 + G(y))$

Example 1 (LMI in Standard Form).

Characterize the constraint set: $F(x) = \begin{bmatrix} \boxed{x_1 + x_2} & x_2 + 1 \\ x_2 + 1 & x_3 \end{bmatrix} \succeq 0$ → P.S. 1).

Decision variable: x_1, x_2, x_3 $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

1°: Is this a LMI? $F(x) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F_0} + x_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{F_1} + x_2 \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{F_2} + x_3 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{F_3}$

2°: $F(x) \succeq 0 \iff$

$$\boxed{\begin{aligned} x_1 + x_2 &\geq 0 \\ x_3 &\geq 0 \\ (x_1 + x_2)x_3 - (x_2+1)^2 &\geq 0 \end{aligned}}$$

$\Leftarrow$ Linear constr~~aint~~ ! nonlinear

Convex constraint? Yes

↳ nonlinear / convex constraint !!

- Generalize it to matrix variable.

$X \in \mathbb{R}^{2\times 2}$, decision variable $\quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$

$F: \mathbb{R}^{2\times 2} \to \mathbb{R}^{4\times 4}$, $\quad F(X) = \left[\begin{array}{c|cc} X & \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & X + I_{2\times 2} \end{array}\right] \succeq 0$

Is this a LMI ?

$$F(X) = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}}_{F_0} + x_{11} \cdot \underbrace{\begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}}_{F_1} + x_{12} \cdot \underbrace{\begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix}}_{F_2} + x_{22} \underbrace{\begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix}}_{F_3}$$

$$= F_0 + G(X)$$

$G(\alpha X) = \alpha G(X)$

$X \in \mathbb{R}^{2\times 3}$

$F(X) \triangleq \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 \end{bmatrix} \succeq 0$

Is this linear ?

$F(\alpha X) \neq \alpha F(X)$

Linear Matrix Inequalities (3/4)

- General Linear Matrix Inequalities (LMI)
  - Let $\mathcal{X}$ be a finite-dimensional real vector space.

    例) $\mathbb{R}^n$, or $\mathbb{R}^{n\times n}$.
  - $F : \mathcal{X} \to \mathcal{S}^n$ is an *affine* mapping from $\mathcal{X}$ to $n \times n$ symmetric matrices
  - Then $F(X) \succeq 0$ is called also an LMI in variable $X \in \mathcal{X}$
  - Translation to standard form: Choose a basis $X_1, \ldots, X_m$ of $\mathcal{X}$ and represent $X = x_1 X_1 + \cdots + x_m X_m$ for any $X \in \mathcal{X}$. For a given affine mapping $F : \mathcal{X} \to \mathcal{S}^n$, we can define $\hat{F} : \mathbb{R}^m \to \mathcal{S}^n$ as

$$\hat{F}(x) \triangleq F(X) = F(0) + \sum_{i=1}^{m} x_i [F(X_i) - F(0)]$$

where x is the coordinate of X w.r.t. the basis $X_1, \ldots, X_m$.

Linear Matrix Inequalities (4/4)

Example 2.

Find conditions on matrix P to ensure that $V(x) = x^T P x$ is a Lyapunov function for a linear system $\dot{x} = Ax$

· $V(x)$ is Lyapunov func if $\begin{cases} ① \ V \text{ is P.D.} \leftarrow \text{"observable"} \\ ② \ \dot{V} \text{ is N.D.} \leftarrow \end{cases}$

$(\nabla V)^T (Ax)$

$\Rightarrow ① \Rightarrow \boxed{P \text{ is P.D.}}$ ①

$② \Rightarrow x^T(A^T P + PA)x < 0 \ \forall x \Rightarrow \boxed{A^T P + PA \prec 0}$ ②

N.D

① and ② are conditions in P, they are LMI for P.

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, $E^{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow P = \sum\sum p_{ij} E^{ij}$

$= p_{11} E^{11} + p_{12} E^{12} + p_{22} E^{22}$

$\Rightarrow \left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Rightarrow \begin{cases} \sum\sum p_{ij} E^{ij} \succ 0 \\ -\sum\sum p_{ij} (A^T E^{ij} + E^{ij} A) \succ 0 \end{cases}$

Schur Complement Lemma (1/2)

Lemma 1 (Schur Complement Lemma).

Define $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. *The following three sets of inequalities are equivalent.*

$$M \succ 0 \quad \Leftrightarrow \quad \begin{cases} A \succ 0 \\ C - B^T A^{-1} B \succ 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} C \succ 0 \\ A - BC^{-1}B^T \succ 0 \end{cases}$$

- **Proof:** The lemma follows immediately from the following identities:

$$T^T \begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix} \overbrace{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}}^{M} \overbrace{\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}}^{T} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I & -BC^{-1} \\ 0 & I \end{bmatrix}}\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ -C^{-1}B^T & I \end{bmatrix}} = \begin{bmatrix} A - BC^{-1}B^T & 0 \\ 0 & C \end{bmatrix}$$

Schur Complement Lemma (2/2)

- The proof of Schur complement lemma also reveals more general relations between the numbers of negative, zero, positive eigenvalues of
  - M vs. A and $C - B^T A^{-1} B$
  - M vs. C and $A - BC^{-1} B^T$
- Schur complement lemma is a very useful result to transform nonlinear (quadratic or bilinear) matrix inequalities to linear ones.

e.g. $X \in \mathbb{R}^{2\times 3}$, $\underset{2\times 3}{X}\,\overset{3\times 2}{X^T} \preceq I_{2\times 2}$ $\longrightarrow$ Not LMI $F(X) = XX^T$

$\Rightarrow XX^T - I \preceq 0 \Rightarrow I - XX^T \succeq 0$

By schur complement Lemma $\Leftrightarrow \underbrace{\overbrace{\begin{bmatrix} I & X^T \\ X & I \end{bmatrix}}^{M} \succeq 0}_{\text{LMI}}$

Outline

- Linear Matrix Inequalities
- Semidefinite Programming Problems
- S-Procedure
- Some Examples
- Conclusion

Semidefinite Programming (1/3)

- **Semidefinite Programming (SDP) Problem**: Optimization problem with linear objective, and Linear Matrix Inequality and linear equality constraints:

$$\begin{cases} \text{minimize:} & c^T x \\ \text{subject to:} & F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0 \\ & Ax = b \end{cases} \tag{1}$$

linear func of $x \in \mathbb{R}^n$

Constraint set is convex

- Linear *equality* constraint in (1) can be eliminated. So essentially SDP can be viewed as optimizing linear function subject to only LMI constraints.

- SDP is a particular class of convex optimization problem. Global optimal solution can be found efficiently.

- Optimizing nonlinear but convex cost function subject to LMI constraints is also a convex optimization that can often be solved efficiently.

Semidefinite Programming (2/3)

Standard forms of SDP in *matrix variable*:

$X \in \mathbb{R}^{n\times n}$

$\in S^n$

- **SDP Standard Prime Form**:

$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

e.g. min $x_{11} + 2x_{12}$

subj: $x_{11} + x_{22} = 2$

$\begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq 0$

$C \cdot X = tr(C^T X)$

$= \sum_i \sum_j C_{ij} X_{ij}$

$$\begin{cases} \min\limits_{X \in \mathcal{S}^n} : & f_p(X) = C \bullet X \\ \text{subject to:} & A_i \bullet X = b_i, i = 1, \ldots, m \\ & X \succeq 0 \end{cases} \quad (2)$$

min $f_p(x)$

sub $g(x) \le 0$

- **SDP Dual form**:

$$\begin{cases} \max\limits_{y \in \mathbb{R}^m} : & f_d(y) = b^T y \\ \text{subject to:} & \sum_{i=1}^m y_i A_i \preceq C \end{cases} \quad (3)$$

HW

- One can derive the dual from the prime using either standard Lagrange duality method or more specialized Fenchel duality results
- The dual form (3) is equivalent to (1) (after eliminating the equality constraint $Ax = b$ in (1))

Semidefinite Programming (3/3)

$A \succeq 0,\ B \succeq 0 \Rightarrow tr(A^T B) \geq 0$

- **SDP Weak Duality**: $f_p(X) \geq f_d(y)$ for any primal and dual feasible X and y ; X is feasible , y is feasible for (3)

  due to PSD cone is acute.

  $$f_p(X) - f_d(y) = C \bullet X - b^T y = C \bullet X - \sum_i y_i (A_i \bullet X) = \underbrace{\left(C - \sum_i y_i A_i\right)}_{\succeq 0} \bullet \underbrace{X}_{\succeq 0} \geq 0$$

- **SDP Strong Duality**: $f_p(X^*) = f_d(y^*)$ holds under Slater's condition:

  $\exists X \succ 0 , \quad A_i \bullet X = b_i$

- Many control and optimization problem can be formulated or translated into SDP problems

- Various computationally difficult optimization problems can be effectively approximated by SDP problems (SDP relaxation...)

- We will see some examples after introducing an important technique: *S-procedure*

Outline

- Linear Matrix Inequalities
- Semidefinite Programming Problems
- **S-Procedure**
- Some Examples
- Conclusion

S-Procedure (1/2)

- Many stability/engineering problems require to certify that a given function is sign-definite over certain subset of the space

- Mathematically, this condition can be stated as follows:

$$g_0(x) \geq 0 \quad \text{on} \quad \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \tag{4}$$

- Given functions $g_0, \ldots, g_m$, we want to know whether the condition holds. Sometimes we may also want to find a g_0 satisfying this condition for given $g_1, \ldots, g_m$.

- Conservative but useful condition: $\exists$ PSD functions $s_i(x)$ s.t.

$$g_0(x) - \sum_i s_i(x) g_i(x) \geq 0, \forall x \in \mathbb{R}^n$$

This is the so-called **Generalized S-Procedure**

S-Procedure (2/2)

Now consider an important special case: $g_i(x) = x^T G_i x, i = 0, 1, \ldots$ are quadratic functions

- Requirement (4) becomes:

$$\forall x \in \mathbb{R}^n, \quad x^T G_1 x \geq 0, \ldots, x^T G_k x \geq 0 \quad \Rightarrow \quad x^T G_0 x \geq 0$$

- Sufficient condition (S-procedure): $\exists \alpha_1, \ldots, \alpha_m \geq 0$ with

$$G_0 \succeq \alpha_1 G_1 + \cdots + \alpha_m G_m$$

- S-Procedure is lossless if $m = 1$ and $\exists \hat{x}$ s.t. $\hat{x}^T G_1 \hat{x} > 0$ (constraint qualification)

Outline

Some Examples (1/4)

Example 3 (Eigenvalue Optimization).

Given symmetric matrices $A_0, A_1, \ldots, A_m$. Let $S(w) = A_0 + \sum_i w_i A_i$. Find weights $\{w_i\}_{i=1}^m$ to minimize $\lambda_{\max}(S(w))$

symmetric

$$\boxed{\min_{w} \lambda_{\max}(S(w))} \Leftarrow \text{nonlinear optimization}$$

$\Rightarrow f(w)$

Recall: $\lambda_{\max}(S(w)) \leq \beta \iff S(w) \preceq \beta I \iff S(w) - \beta I \preceq 0$

$$\iff \begin{cases} \min_{w,\beta} \beta \\ \text{subj to: } \underbrace{S(w) - \beta I \preceq 0} \end{cases} \quad \text{SDP}$$

$$A_0 + w_1 A_1 + \ldots + w_m A_m - \beta I \preceq 0$$

LMI in (w, β)

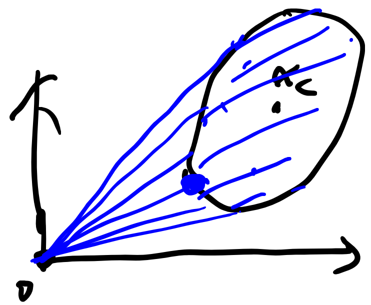
Some Examples (2/4)

Example 4 (Ellipsoid inequality).

Given $R \in \mathcal{S}^n_{++}$, the set $E = \{x \in \mathbb{R}^n : (x - x_c)^T R (x - x_c) \le 1\}$ is an ellipsoid with center x_c. Find the point in E that is the closet to the origin.

Problem:

$$\begin{cases} \min_x \|x\|^2 \\ \text{subj to: } (x - x_c)^T R (x - x_c) \le 1 \end{cases}$$



$\Leftrightarrow 1 - (x - x_c)^T R (x - x_c) \succeq 0$

Schur complement lemma:

$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0$

$\Leftrightarrow \begin{cases} A \succ 0 \\ C - B^T A^{-1} B \succ 0 \end{cases}$

$C = 1, \quad A^{-1} = R, \quad B = x - x_c$

$\Leftrightarrow \begin{bmatrix} R^{-1} & x - x_c \\ (x - x_c)^T & 1 \end{bmatrix} \succeq 0$ LMI in x

Some Examples (3/4)

Example 5 (Linear Feedback Control Gain Design).

Given a linear control system $\dot{x} = Ax + Bu$ with linear state feedback $u = Kx$. Find K to stabilize the system

- cl-system: $\dot{x} = Ax + B(Kx) = \underbrace{(A+BK)}_{A_{cl}}x$ stable $\iff \exists P \succ 0$ such

$(A+BK)^T P + P(A+BK) \prec 0$

$\Downarrow$

$V(x)$ is Lyapunov func for cl-sys

we know, for linear sys: asym stable $\iff$ exp. stable

Lyapunov func. exp stability $\Leftarrow$ $\boxed{\dot{V} \le -\alpha V}$ $\alpha > 0$

$V(x) = x^T P x$

$\iff x^T(A_{cl}^T P + P A_{cl})x \le -\alpha(x^T P x)$

$\iff A_{cl}^T P + P A_{cl} + \alpha P \preceq 0 \iff (A+BK)^T P + P(A+BK) + \alpha P \preceq 0$ $F(\alpha, P, K)$

variable (P, K, α)

$\iff P^{-1} F(\alpha, P, K)(P^{-1})^T \preceq 0$

$\iff P^{-1}A^T + P^{-1}K^T B^T + AP^{-1} + BKP^{-1} + \alpha P^{-1} \preceq 0$

$P^{-1} \triangleq X$, $KP^{-1} \triangleq Y$ ($P^{-1}K^T = Y^T$)

$\Leftarrow g(\alpha, X, Y)$

$$\min_{\alpha, X, Y} \; -\alpha$$

$g(\alpha, X, Y) \preceq 0$

$\alpha > 0$

$X \succ 0$,

SDP

Some Examples (4/4)

Example 6 (Robust Stabiltiy).

Given system $\dot{x} = Ax + u$ with uncertain feedback $u = g(x)$. Suppose all we know is that the feedback law satisfies: $\|g(x)\|^2 \leq \beta\|x\|^2$. Find Lyapunov function $V(x) = x^T P x$ to ensure exponential stability.

Concluding Remarks

- Linear matrix inequalities impose convex constraints

- Semidefinite programming problem: optimize linear cost subject to LMI constraints

- SDP has broad applications in various engineering fields: signal processing, networking, communication, control, machine learning, big data...

References

More Discussions

-

More Discussions

-