MEE5114 (Sp22) Advanced Control for Robotics

Lecture 2: Rigid Body Configuration and Velocity

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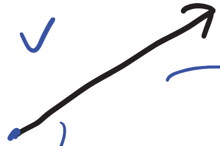
Outline

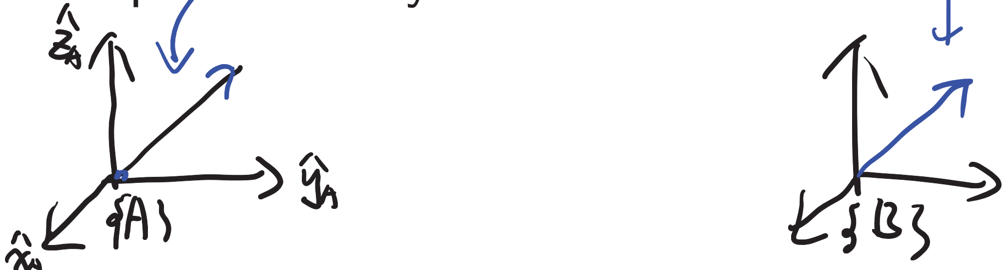
- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

Free Vector

- **Free Vector**: geometric quantity with length and direction



- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented by its coordinates v in the reference frame.



- v denotes the physical quantity while ${}^{A}v$ denote its coordinate wrt frame $\{A\}$.

Frame : coordinate sys based on basis vectors

$\{A\}$- frame: $\{\hat{x}_A, \hat{y}_A, \hat{z}_A\}$.

$${}^{A}\hat{x}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

${}^{A}v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ means

$v = 1 \cdot x_A + 2\, y_A + 3\, z_A$

Point

- **Point**: p denotes a point in the physical space

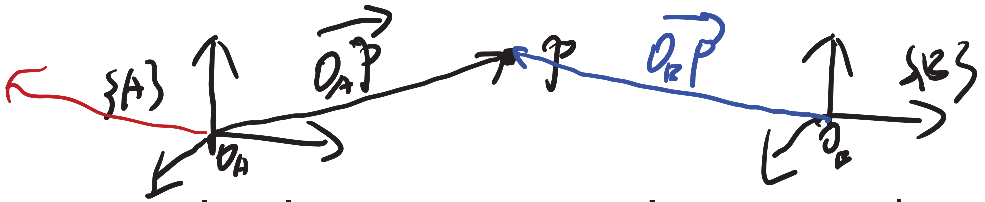
$\bullet\ p$

- A point p can be represented by a vector from frame origin to p

${}^A(\overrightarrow{O_B p})$

- ${}^A p$ denotes the coordinate of a point p wrt frame $\{A\}$

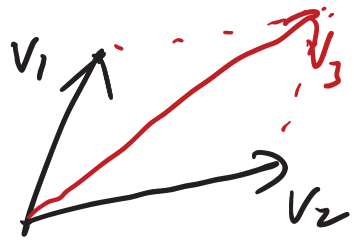
${}^A p = {}^A(\overrightarrow{O_A p})$



${}^B p = {}^B\overrightarrow{O_B p}$

- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector for which the reference frame is clear from the context. think in "coordinate-free" way whenever possible

e.g.



- coordinate-free: $v_3 = v_1 + v_2$
- But also express "physics" in different frames
- ${}^A v_3 = {}^A v_1 + {}^A v_2$; ${}^B v_3 = {}^B v_1 + {}^B v_2$

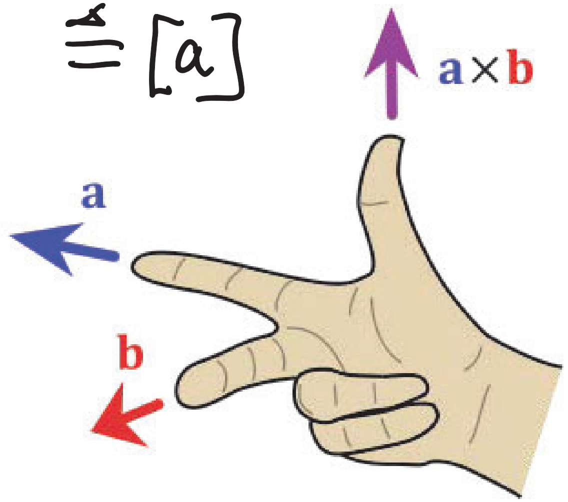
Cross Product

${}^A v_3 \times {}^B v_1 + {}^B v_2$

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\triangleq [a]} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (1)$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



Properties:

- $\|a \times b\| = \|a\| \|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$

Skew symmetric representation

$A = A^T$

- It can be directly verified from definition that $a \times b = [a]b$, where

$$[a] \triangleq \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}} \quad (2)$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

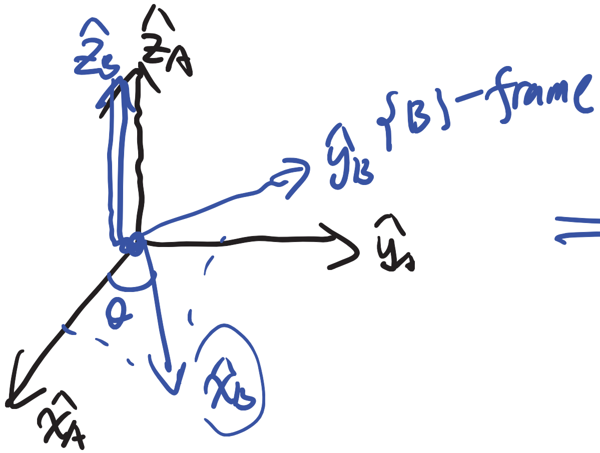
$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $[a] = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

- $[a] = -[a]^T$ (called skew symmetric) $\Leftrightarrow A = -A^T$
- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$ ⇐ (right hand rule)
- **Rotation Matrix**: specifies orientation of one frame relative to another

$$ {}^{A}R_{B} \triangleq \begin{bmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} & {}^{A}\hat{z}_{B} \end{bmatrix} $$



$$ \Longrightarrow {}^{A}R_{B} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

- A valid rotation matrix R satisfies: $\boxed{\text{(i) } R^T R = I; \text{ (ii) } \det(R) = 1}$

$$ \begin{bmatrix} {}^{A}\hat{x}_{B}^{T} \\ {}^{A}\hat{y}_{B}^{T} \\ {}^{A}\hat{z}_{B}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} & {}^{A}\hat{z}_{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ \det({}^{A}R_{B}) = {}^{A}\hat{x}_{B}^{T} \left({}^{A}\hat{y}_{B} \times {}^{A}\hat{z}_{B} \right) $$

Special Order Orthogonal Group

$A \in SO(3)$ $= 1$

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

$SO(3)$

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a, b, c \in G$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Use of Rotation Matrix (1/2)

- Representing an orientation ${}^{A}R_B$ ⇐ from definition

- Changing the reference frame $\boxed{{}^{A}R_B}$: Given vector v, it's coordinates in $\{A\}$, $\{B\}$, are ${}^{A}v$, ${}^{B}v$

$\boxed{{}^{A}v = {}^{A}R_B \, {}^{B}v}$: "coordinate - free" proof

— same physical vector v, suppose ${}^{A}v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, ${}^{B}v = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

"physics"

$$\Rightarrow \quad v = \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A$$

$$v = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B$$

$$\Rightarrow \underline{\alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A} = \underline{\beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B} \quad \text{— "physics"}$$

"state this physics" in $\{A\}$-frame $\Rightarrow \alpha_1 {}^{A}\hat{x}_A + \alpha_2 {}^{A}\hat{y}_A + \alpha_3 {}^{A}\hat{z}_A = \beta_1 {}^{A}\hat{x}_B + \beta_2 {}^{A}\hat{y}_B + \beta_3 {}^{A}\hat{z}_B$

Use of Rotation Matrix (2/2)

$$\underbrace{\left[{}^A\hat{x}_A \;\vdots\; {}^A\hat{y}_A \;\vdots\; {}^A\hat{z}_A \right]}_{I} \cdot \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{{}^A v} = \underbrace{\left[{}^A\hat{x}_B \;\vdots\; {}^A\hat{y}_B \;\vdots\; {}^A\hat{z}_B \right]}_{{}^A R_B} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{{}^B v}$$

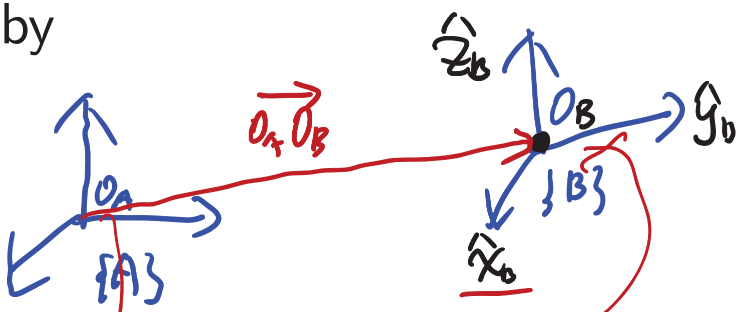
- Rotating a vector or a frame $\mathrm{Rot}(\hat{\omega}, \theta)$: will be discussed in next lecture.

"action": verb   operator view

Rigid Body Configuration

different orientation

- Given two coordinate frames {A} and {B}, the configuration of B relative to A is determined by
  - $^A R_B$ and $^A o_B$



$$^A R_B = \left[{}^A\hat{x}_B \quad {}^A\hat{y}_B \quad {}^A\hat{z}_B \right]$$

$$^A o_B$$

- For a (free) vector r, its coordinates $^A r$ and $^B r$ are related by:

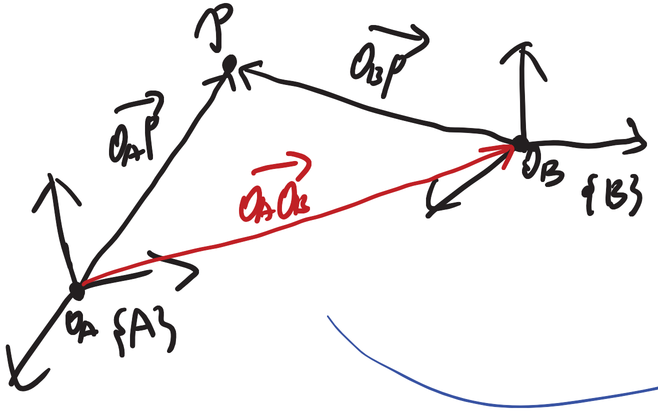
$$\underbrace{{}^A r = {}^A R_B \, {}^B r}_{\text{Linear relation}} \Longrightarrow$$

$${}^A\tilde{r} = \begin{bmatrix} {}^A r \\ 1 \end{bmatrix}, \quad {}^B\tilde{r} = \begin{bmatrix} {}^B r \\ 1 \end{bmatrix}$$

$${}^A\tilde{r} = {}^A T_B \, {}^B\tilde{r}$$

$$\Rightarrow {}^A\tilde{r} = {}^A o_B + {}^A R_B \, {}^B\tilde{r}$$

- For a point p, its coordinates $^A p$ and $^B p$ are related by:



"coordinate-free"

$$\Rightarrow \underbrace{\overrightarrow{O_A p} = \overrightarrow{O_A O_B} + \overrightarrow{O_B p}}$$

choose "A" frame to express "physics"

$${}^A p = {}^A o_B + {}^A R_B \, {}^B p$$

$${}^A\tilde{r} = \begin{bmatrix} {}^A r \\ 0 \end{bmatrix}$$

$${}^A\overrightarrow{O_A p} = {}^A\overrightarrow{O_A O_B} + {}^A\overrightarrow{O_B p}$$

$$= {}^A R_B \, {}^B p$$

Homogeneous Transformation Matrix

- **Homogeneous Transformation Matrix**: ${}^AT_B$

$$\underset{\uparrow}{{}^Ap} = \underbrace{{}^AO_B}_{\in R^3} + \underbrace{{}^AR_B}_{3\times 3} \cdot \underset{\uparrow \in \mathbb{R}^3}{{}^Bp} \iff \begin{bmatrix} {}^Ap \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^AR_B & \overset{\mathbb{R}^3}{{}^AO_B} \\ \underbrace{0}_{1\times 3} & \underbrace{1}_{1\times 1} \end{bmatrix}}_{4\times 4} \begin{bmatrix} {}^Bp \\ 1 \end{bmatrix}$$

affine relation

$${}^AT_B \triangleq \begin{bmatrix} {}^AR_B & {}^AO_B \\ 0 & 1 \end{bmatrix}$$

$T = (R, p)$, configuration of $\{B\}$ relative to $\{A\}$

- Homogeneous coordinates:

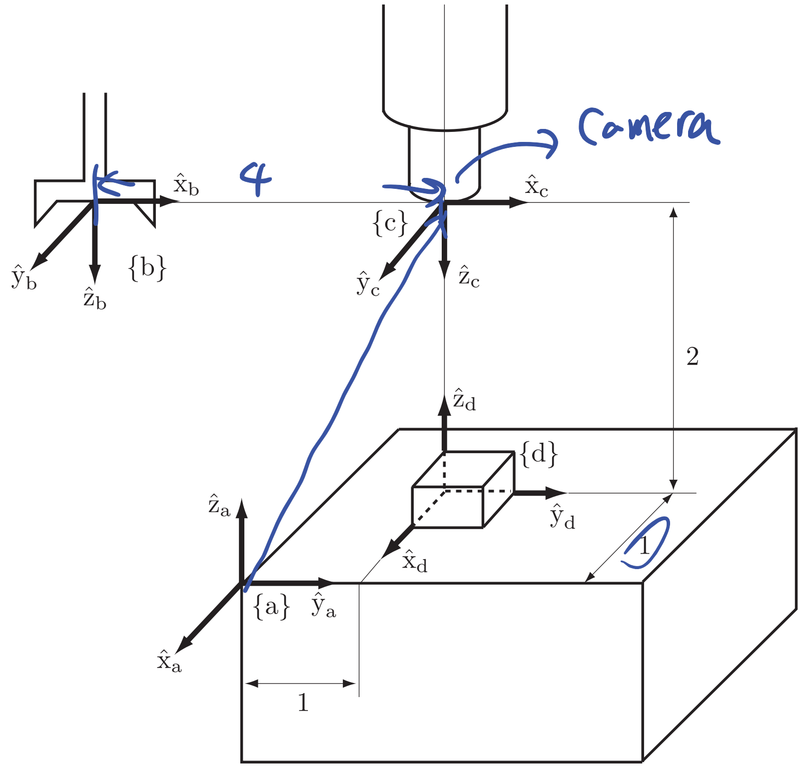
Given a point $p \in \mathbb{R}^3$, its homogeneous coordinate is given by $\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix} \in \mathbb{R}^4 \Rightarrow \boxed{{}^A\tilde{p} = {}^AT_B \cdot {}^B\tilde{p}}$

Given vector v, its homogeneous coordinate is $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$v = p_1 - p_2 \qquad \tilde{v} = \tilde{p}_1 - \tilde{p}_2$

Example of Homogeneous Transformation Matrix

Fixed frame $\{a\}$; end effector frame $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$. Suppose $\|p_c - p_b\| = 4$



1 Camera "Location"? ${}^aR_c$, ${}^ap_c$

$${}^aT_c = ({}^aR_c, {}^ap_c)$$

$${}^aR_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad {}^ap_c = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{\hat{x} = 0 \cdot \hat{x}_a + 1 \cdot \hat{y}_a + 0 \cdot \hat{z}_a}$$

$${}^aT_c = \begin{bmatrix} {}^aR_c & {}^ap_c \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

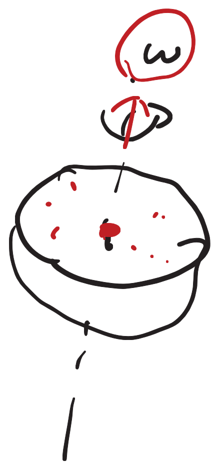
2 . end-effector frame : ${}^aT_b$,

$${}^aT_b = {}^aT_c \, {}^cT_b$$

$${}^cT_b = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

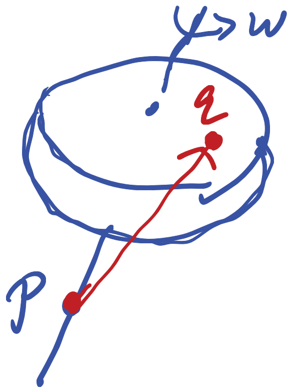
Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion



Rigid Body Velocity (1/2)

- Consider a rigid body with angular velocity: ω (this is a free vector).
- Suppose the actual rotation axis passes through a point p; Let v_p be the velocity of the point p.



v_q

$v_q = v_p + \omega \times (\overrightarrow{pq})$

Question: A rigid body contains infinitely many points with different velocities. How to parameterize all of their velocities?

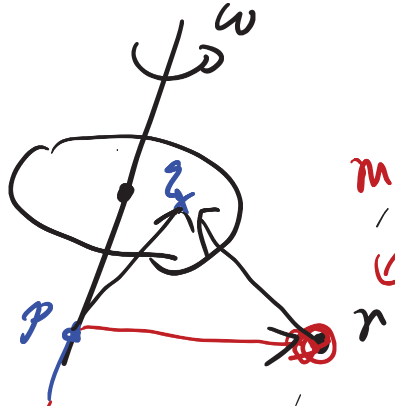
- Consider an arbitrary body-fixed point q (means that the point is rigidly attached to the body, and moves with the body), we have:

$$\boxed{v_q = v_p + \omega \times (\overrightarrow{pq})} \tag{3}$$

- The velocity of an arbitrary body-fixed point depends only on (ω, v_p, p) and the location of the point q.

Rigid Body Velocity (2/2)

- Fact: The representation form (3) is independent of the reference point p.



may not be body-fixed

- Consider an arbitrary point r in space
  - r may not be on the rotation axis
  - r may be a stationary point in space (does not move)
  - Let v_r be the **velocity of the body-fixed point currently coincides with** r

    rigidly attached to the body
  - We still have: $\boxed{v_q = v_r + \omega \times (\overrightarrow{rq})}$

$$V_q = V_p + \omega \times (\overrightarrow{pq}), \qquad V_r = V_p + \omega \times (\overrightarrow{pr})$$

$$= V_r - \omega \times \overrightarrow{pr} + \omega \times (\overrightarrow{pq})$$

$$= V_r + \omega \times (\overrightarrow{pq} - \overrightarrow{pr}) = V_r + \omega \times \overrightarrow{rq}$$

- The body can be regarded as translating with a linear velocity v_r, while rotating with angular velocity ω about an axis passing through r

Rigid Body Velocity: Spatial Velocity (Twist)

Featherstone Murray

- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\omega, v_r) \in \mathbb{R}^6$
  - ω: angular velocity
  - v_r: velocity of the body-fixed point currently coincides with r
  - For any other body-fixed point q, its velocity is

$$v_q = v_r + \omega \times (\overrightarrow{rq})$$

- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point r
- A rigid body with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at v_r while rotating with angular velocity ω about an axis passing through r
  - This is just one way to interpret the motion.

Spatial Velocity Representation in a Reference Frame

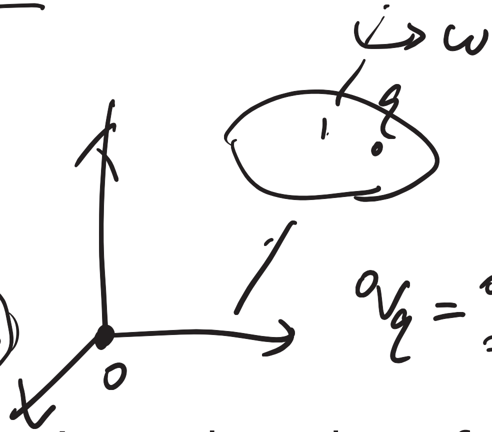
- Given frame $\{O\}$ and a spatial velocity $\mathcal{V}$

- Choose o (the origin of $\{O\}$) as the reference point to represent the rigid body velocity

- Coordinates for the $\mathcal{V}$ in $\{O\}$:

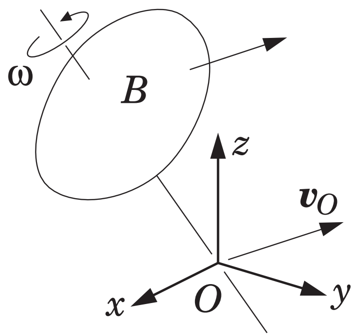
$A\mathcal{V}$

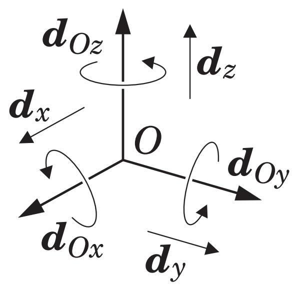
$${}^O\mathcal{V}_o = ({}^O\omega, {}^O v_o)$$



$${}^O v_q = {}^O v_o + {}^O\omega \times {}^O q$$

- By default, we assume the origin of the frame is used as the reference point: ${}^O\mathcal{V} = {}^O\mathcal{V}_o$





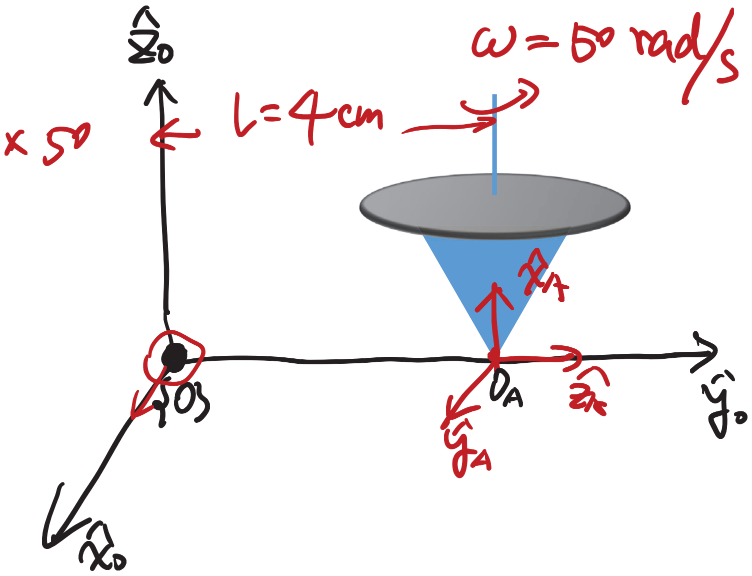
Example of Twist I

- Example I: What's the twist of the spinning top?

choose $\{O\}$-frame. $\quad {}^0\mathcal{V}_{top} = \begin{bmatrix} {}^0\omega \\ \hline {}^0v_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \text{ rad/s} \\ \hline 2 \\ 0 \\ 0 \end{bmatrix}$

0.04×50

$= 2$



choose $\{A\}$-frame: $\quad {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_{O_A} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{array}{cc} R & p \end{array}$$

$${}^0T_A = \begin{bmatrix} {}^0R_A & {}^0p_A \\ \hline 0 & 1 \end{bmatrix} \Rightarrow {}^0X_A = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

V.F.Y.

$$\underbrace{{}^0\mathcal{V}_{top}}_{\mathbb{R}^6} = \underbrace{{}^0X_A}_{6\times 6} \; \underbrace{{}^A\mathcal{V}_{top}}_{\in \mathbb{R}^6}$$

Example of Twist II

- Example II: what's ${}^{b}\mathcal{V}_{car}$ ${}^{s}\mathcal{V}_{car}$

Car rotates about $\hat{\omega}$

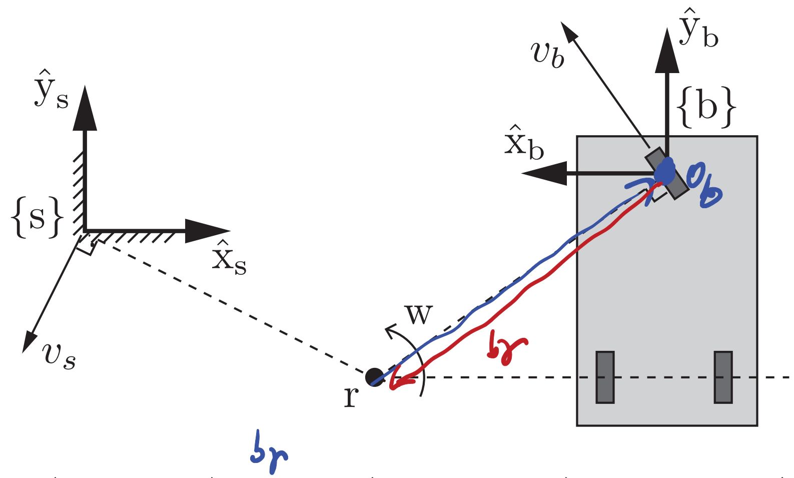
$${}^{b}\mathcal{V}_{car} = \begin{bmatrix} {}^{b}\omega \\ {}^{b}v_{O_b} \end{bmatrix}, \quad {}^{b}\omega = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad {}^{b}v_{O_b} = {}^{b}\omega \times {}^{b}\overrightarrow{rO_b} = (-{}^{b}r)$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \\ \hline 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$${}^{s}\mathcal{V}_{car} = \begin{bmatrix} {}^{s}\omega \\ {}^{s}v_{O_s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}$$

$${}^{s}v_{O_s} = {}^{s}\omega \times {}^{s}(\overrightarrow{rs}) = {}^{s}\omega \times (-{}^{s}r) = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$



$r_s = (2, -1, 0)$, $\cancel{r_b}$ ${}^{b}r = (2, -1.4, 0)$, w=2 rad/s

Change Reference Frame for Twist (1/2)

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames $\{A\}$ and $\{B\}$

$$ {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix} $$

- They are related by ${}^A\mathcal{V} = {}^AX_B\,{}^B\mathcal{V}$

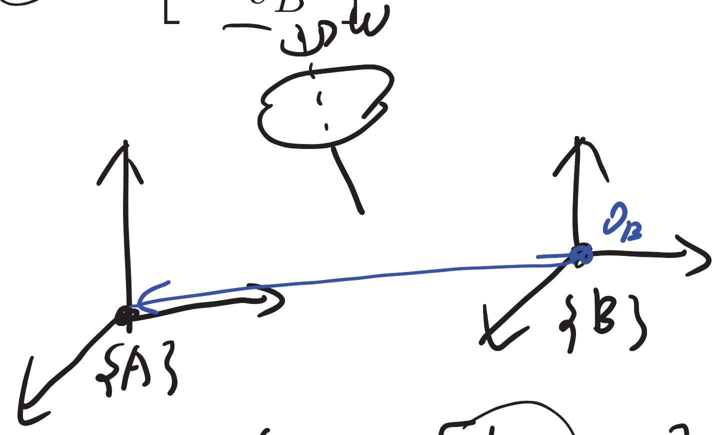
① ${}^A\omega = {}^AR_B\,{}^B\omega \Longrightarrow {}^A\omega = \begin{bmatrix} {}^AR_B & 0 \end{bmatrix} \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}$

② "coordinate-free"

v_A: body-fixed pt coincides with O_A velocity

v_B: . . . . O_B

$$ v_A = v_B + \omega \times (\overrightarrow{BA}) $$



$$ {}^AT_B = \begin{bmatrix} {}^AR_B & {}^Ap_B \\ 0 & 1 \end{bmatrix} $$

$\Longrightarrow$ choose $\{A\}$ to express "physics"

$$ {}^Av_A = {}^Av_B + {}^A\omega \times {}^A(\overrightarrow{BA}) $$

${}^A(\overrightarrow{BA}) = {}^A(-\overrightarrow{AB}) = -{}^AO_B$

$$ = {}^AR_B\,{}^Bv_B + {}^AR_B\,{}^B\omega \times (-{}^AO_B) $$

$$ = {}^AR_B\,{}^Bv_B + {}^AO_B \times ({}^AR_B\,{}^B\omega) $$

$[{}^AO_B]$

Change Reference Frame for Twist (2/2)

$$\Rightarrow {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_A \end{bmatrix} = \underbrace{\begin{bmatrix} {}^AR_B & 0 \\ [{}^Ao_B]\,{}^AR_B & {}^AR_B \end{bmatrix}}_{6\times 6 \text{ matrix}} \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}$$

$$\triangleq {}^AX_B$$

$$= \begin{bmatrix} [{}^Ao_B]\,{}^AR_B & {}^AR_B \end{bmatrix} \overbrace{\begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}}^{{}^B\mathcal{V}}$$

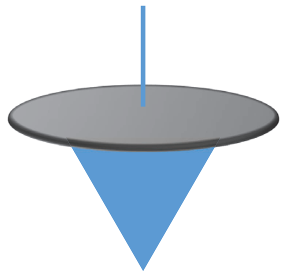
$${}^AT_B \triangleq (R, p) \Rightarrow {}^AX_B = \underbrace{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}}$$

- If configuration {B} in {A} is $T = (R, p)$, then

$${}^AX_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

$$\text{Given } T = (R, p) \xrightarrow{[\mathrm{Ad}_T]} X =$$

Example I Revisited



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- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

- recall: • linear velocity $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\|v\| \cdot \hat{v}} = \sqrt{5} \cdot \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

scalar

- angular velocity : $\omega = \hat{\omega} \cdot \dot{\theta}$

scalar

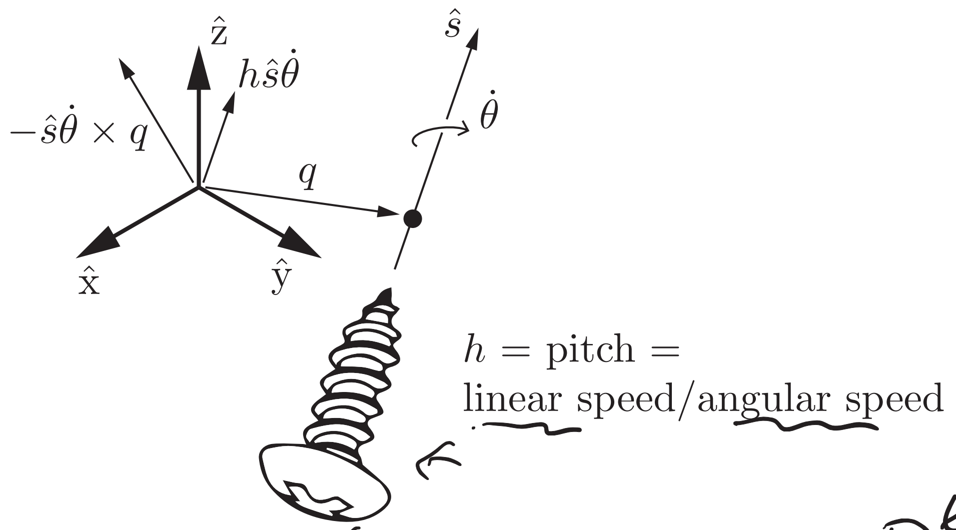
rotation axis (unit vector)

• rigid-body velocity: twist $\mathcal{V} = (v, \omega)$

Screw Motion: Definition

: standard/canonical motion for rigid body motion

- Rotating about an axis while also translating along the axis



$h\dot{\theta}$

- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

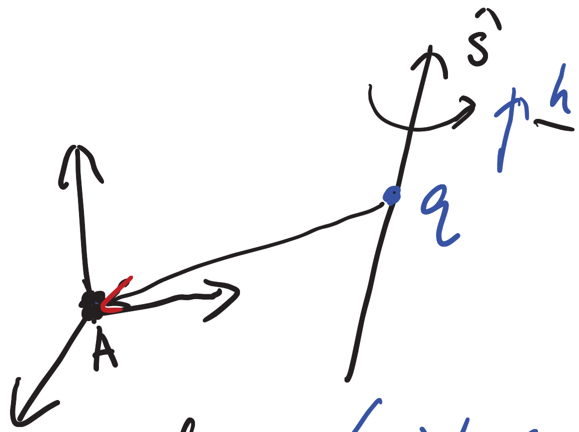
From Screw Motion to Twist

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$
- Fix a reference frame $\{A\}$ with origin o_A.
- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^Av_{o_A})$

$${}^A\omega = {}^A\hat{s}\cdot\dot{\theta}$$

$${}^Av_{o_A} = (h\dot{\theta})\,{}^A\hat{s} + {}^A\omega \times (-{}^Aq)$$

$$= {}^A\hat{s}(h\dot{\theta}) - {}^A\omega \times {}^Aq$$



coordinate-free : / pick q as reference point

$v_{o_A} \quad v_q + \omega \times (\overrightarrow{q o_A})$

$(h\dot{\theta})\cdot\hat{s} + \omega \times (\overrightarrow{q o_A})$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$\to \omega$

$$\omega = \hat{s}\dot{\theta} \qquad v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

From Twist to Screw Motion

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\dot{\theta}$
  - If $\omega = 0$, then it is a pure translation $(h = \infty)$

pure linear

$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

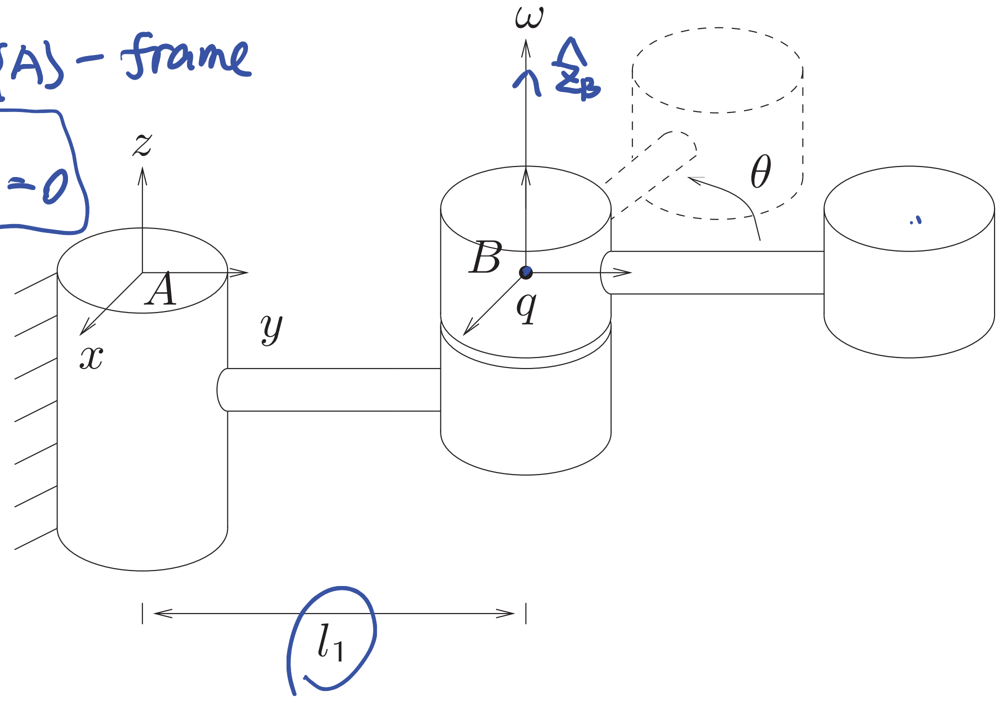
  - If $\omega \neq 0$:

$$\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}$$

you can ~~pu~~ plug into the eqn on previous slide to very the result.

Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$? choose {A}-frame



· screw axis: $\boxed{{}^A\hat{S} = {}^A\hat{z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^Aq = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, \; h = 0}$

$\dot{\theta} = 2$

$${}^A\omega = {}^A\hat{S}\dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$${}^Av_{0_A} = 0 - {}^A\omega \times {}^Aq = -\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2l_1 \\ 0 \\ 0 \end{bmatrix}$$

- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

$w = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$, use the previous formula

$$\Rightarrow q = \frac{w \times v}{\|w\|^2} = \#,$$

Screw Representation of a Twist

$\boxed{\omega = \hat{\omega}\dot{\theta}}$

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\boxed{\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}}$$

spatial velocity (ω, v)

  - In this notation, we think of $\hat{\mathcal{S}}$ as the twist associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

More Discussions