MEE5114 Advanced Control for Robotics

Lecture 3: Operator View of Rigid-Body Transformation

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Skew Symmetric Matrices $a \in \mathbb{R}^3 \to [a] \in \mathbb{R}^{3\times 3}$

- Recall that cross product is a special linear transformation.
- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n\times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

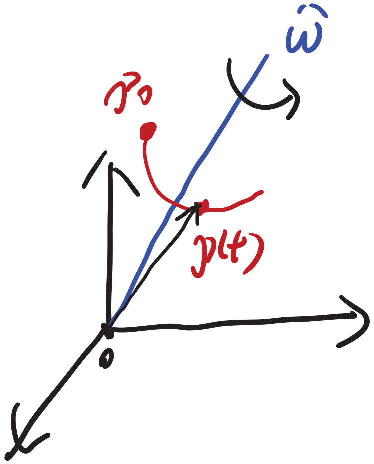
- Note that $\boxed{[\omega] = -[\omega]^T}$ $\leftarrow$ skew symmetric
- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω
- The set of skew-symmetric matrices in: $so(n) \triangleq \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$
- We are interested in case $n = 2, 3$

Rotation matrix $\in SO(3) = \{R^T R = I,$ $\det(R) = 1\}$

Rotation Operation via Differential Equation

- Consider a point initially located at p_0 at time $t = 0$

- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by



$\rightarrow \in \mathbb{R}^{3\times 3}$ $so(3)$

$$\underbrace{\dot{p}(t)}_{\text{linear velocity at time } t} = \boxed{\hat{\omega} \times p(t)} = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \tag{1}$$

op

recall: $\dot{x} = Ax$, $x(0) = x_0$

$\Rightarrow x(t) = e^{At}x_0$

$\dot{p}(t) = [w]p(t) \Rightarrow$ $A = [w]$

- This is a linear ODE with solution: $\boxed{p(t) = e^{[\hat{\omega}]t}p_0}$

rotation operator

- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta}p_0$

- $\mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

The discussion holds for any reference frame.

$R = (\hat{x}, \hat{y}, \hat{z})$

Rotation Matrix as a Rotation Operator (1/3)

theorem

- Every rotation matrix R can be written as $R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

• Any matrix of the form: $e^{[\hat{\omega}]\theta} \in SO(3) = \{ R^T R = I, \det(R) = 1 \}$ $\qquad e^{[\hat{\omega}]0} = e^0 = I$

• $\left(e^{[\hat{\omega}]\theta}\right)^T \left(e^{[\hat{\omega}]\theta}\right) \overset{?}{=} I$

① $\left(e^{[\hat{\omega}]\theta}\right)^T = \left(I + [\hat{\omega}]\theta + \frac{[\hat{\omega}]^2\theta^2}{2!} + \dots \right)^T = \sum_{j=0}^{\infty} \frac{([\hat{\omega}]^T\theta)^j}{j!} = e^{-[\hat{\omega}]\theta}$

② $e^A \cdot e^{-A} = I \Rightarrow \left(e^{[\hat{\omega}]\theta}\right)^T \left(e^{[\hat{\omega}]\theta}\right) = I$

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

- To apply the rotation operation, all the vectors/matrices have to be expressed in the same reference frame (this is clear from Eq (1))

Rotation Matrix as a Rotation Operator (2/3)

- For example, assume $R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}} = \mathrm{Rot}(\hat{\mathrm{x}}; \pi/2)$

- Consider a relation $q = Rp$:
  - **Change reference frame interpretation** : (two frames {A}, {B}, one physical point a)
    - $\underline{R}$: orientation of {B} relative to {A} : i.e. $R = {}^A R_B$
    - then: p, q are coordinates of the same point in {B} and {A}

      $\Rightarrow p = {}^B a$, $q = {}^A a$ $\quad q = Rp \Leftrightarrow \underline{{}^A a = {}^A R_B \, {}^B a}$
  - **Rotation operator interpretation**:

- Have one frame {A}, and two points. $\underline{a, \xrightarrow{Rot(\cdot)} a'}$, $p = {}^A a$, $q = {}^A a'$

${}^A a' = R \cdot {}^A a$

Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame: $R_B = RR_A$

· Have one "frame object", two reference frames

· Frame object: $\{A\}$, orientation in $\{0\}$, is ${}^0R_A$, ${}^BR_A$

$$\Rightarrow {}^0R_A = \underbrace{{}^0R_B}_{R}\,{}^BR_A$$

  - Rotating a frame: $R'_A = RR_A$

· two frame objects

· one refrence frame

$$\{A\} \xrightarrow{R} \{A'\}$$

$R_{A'} = R \cdot R_A$, more precizely, ${}^0R_{A'} = R \cdot {}^0R_A$

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Rotation Matrix Properties $R \in SO(3)$

- $R^T R = I$: definition

- $R_1 R_2 \in SO(3)$, if $R_1, R_2 \in SO(3)$: product of two rotation matrices is also a rotation matrix

- $\|Rp - Rq\| = \|p - q\|$ ← rotation operator preserves distance.

  $p, q \in \mathbb{R}^3$ definition: $\|R(p-q)\|^2 = (p-q)^T \underbrace{R^T R}_{= I} (p-q) = \|p-q\|^2$

- $R(v \times w) = (Rv) \times (Rw)$ ← rotation preserves orientation

- $\boxed{R[w]R^T = [Rw]}$ ← $\dot{x}$

Rotation Operator in Different Frames (1/2)

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.

- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^A\hat{\omega}$ and {B}-frame coordinate ${}^B\hat{\omega}$. We know

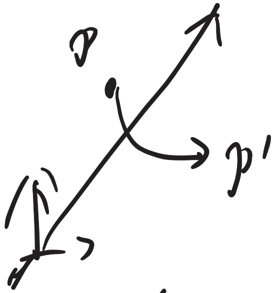
$$ {}^A\hat{\omega} = {}^A R_B \, {}^B\hat{\omega} $$

- Let ${}^B\mathrm{Rot}({}^B\hat{\omega}, \theta)$ and ${}^A\mathrm{Rot}({}^A\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

Rotation Operator in Different Frames (2/2)

- We have the relation:

$$\boxed{{}^A\mathrm{Rot}({}^A\hat{\omega},\theta) = {}^A R_B\, {}^B\mathrm{Rot}({}^B\hat{\omega},\theta)\, {}^B R_A}$$



· Approach 1: two points $p \xrightarrow{\mathrm{Rot}(\cdot)} p' \Rightarrow {}^A p' = \boxed{{}^A\mathrm{Rot}({}^A\hat{\omega};\theta)}\, {}^A p$ $\{A\}$

$\{B\}$-frame: ${}^B p' = \underbrace{{}^B\mathrm{Rot}({}^B\hat{\omega};\theta)\, {}^B p}$

$\Rightarrow {}^A R_B\, {}^B p' = {}^A R_B\, {}^B\mathrm{Rot}(\cdot)\, {}^B p$

${}^A p' = \underbrace{{}^A R_B\, {}^B\mathrm{Rot}(\cdot)\, {}^B R_A}\, {}^A p$

${}^A\mathrm{Rot}({}^A\hat{\omega};\theta) = {}^A R_B\, {}^B\mathrm{Rot}({}^B\hat{\omega};\theta)\, {}^B R_A$

· Approach 2: recall Fact.

for $a \in \mathbb{R}^3$ $[a] \in so(3)$, for any $R \in SO(3)$

$$\boxed{[\underset{\in\mathbb{R}}{Ra}] \in \mathbb{R}^3 \Rightarrow [Ra] = R[a]R^T}$$

$$\mathrm{Rot}({}^A\hat{\omega};\theta) = e^{[{}^A\hat{\omega}]\cdot\theta} = e^{[{}^A R_B\, {}^B\hat{\omega}]\theta} = e^{{}^A R_B [{}^B\hat{\omega}]\, {}^A R_B^T\cdot\theta} = {}^A R_B\, e^{[{}^B\hat{\omega}]\theta}\, {}^B R_A$$

$\Downarrow$ ${}^A\hat{\omega}$

Recall Fact: $e^{PAP^{-1}} = Pe^{A}P^{-1}$, also we know $({}^A R_B)^T = ({}^A R_B)^{-1}$

Outline

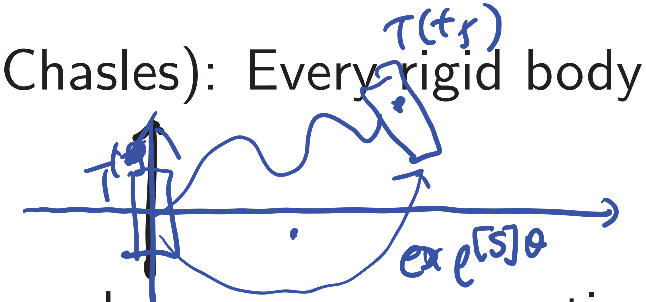
- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Rigid-Body Operation via Differential Equation (1/3)

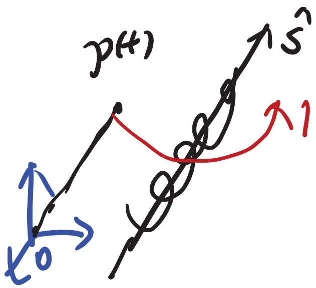
- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(1). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta) p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}] p(t)$ with $p(0) = p_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion



- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed ($\dot\theta = 1$). Let the corresponding twist be $\mathcal{V} = \mathcal{S} = \underbrace{(\omega, v)}_{\mathcal{X}}$. The motion can be described by the following ODE.



$$\boxed{\dot p(t) = \overbrace{\omega \times p(t)}^{[\omega]\cdot p(t)} + v} \Rightarrow \underbrace{\begin{bmatrix} \dot p(t) \\ 0 \end{bmatrix}}_{\dot{\tilde p}(t)} = \begin{array}{c} \begin{array}{cc} 3\times 3 & 3\times 1 \end{array} \\ \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \\ \begin{array}{cc} 1\times 3 & 1\times 1 \end{array} \end{array} \overbrace{\begin{bmatrix} p(t) \\ 1 \end{bmatrix}}^{\tilde p(t)} \in \mathbb{R}^4 \quad (2)$$

$$\dot{\vec{op}}(t) = \underbrace{v_0 + \omega \times \vec{op}(t)}_{\text{definition of twist}}$$

$p(t) \longrightarrow \tilde p(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix}$

$\dot p \longrightarrow \dot{\tilde p}(t) = \begin{bmatrix} \dot p(t) \\ 0 \end{bmatrix}$

- Solution to (2) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\underbrace{\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}}\, t\right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

$\dot{\tilde p}(t) = \tilde A \cdot \tilde p(t)$

$\Rightarrow \tilde p(t) = e^{\tilde A t} \tilde p(0)$

$\tilde p(t) = \boxed{e^{\tilde A t}} \cdot \tilde p(0)$

$\longrightarrow$ rigid body operator

Rigid-Body Operation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation of twist $\mathcal{V}$

$\mathbb{R}^6$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$e^{[S]t} = I + \begin{bmatrix} [\omega] & v \\ 0 & 1 \end{bmatrix} t + \frac{}{2!} \cdots$

$(\hat{s}, q, h)$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$, $[S] = \begin{bmatrix} [\omega] & v \\ 0 & 1 \end{bmatrix}$

- With this notation, the solution to (2) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

$\tilde{p}(0) = e^{[S]0}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

$e^{[S]t} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, $R \in SO(3)$, $p \in \mathbb{R}^3$ $\Leftarrow$ can be proved using definitions of matrix exp.

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t

$\hat{S}$

$se(3)$

$\forall\, \omega \in \mathbb{R}^3 \to [\omega] \in so(3) \xrightarrow{\exp(\cdot)} e^{[\omega]\theta} \in SO(3)$

$\forall\, S \in \mathbb{R}^6 \to [S] \xrightarrow{4\times 4 \text{ matrix}} \in \boxed{se(3)} \xrightarrow{\exp(\cdot)} e^{[S]\theta} \in SE(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$\begin{bmatrix} \omega \\ v \end{bmatrix}$$

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

$$\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree

if not unit speed. $\mathcal{V} = S \cdot \dot{\theta}$

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree

$T = e^{[S]\theta}$

two pts: $\tilde{p} \longrightarrow \tilde{p}'$

more precisely: ${}^{0}\tilde{p}' = {}^{0}T\,{}^{0}\tilde{p}$

- TT_A: "rotate" {A}-frame about $\mathcal{S}$ by θ degree

$T = e^{[S]\theta}$

$p \quad p'$

$\tilde{p}' = T\tilde{p}$

For S, $[Ad_T]S \longrightarrow S'$

For $T \in SE(3)$

- config representation

${}^{A}T_{B}$: config of {B} relative to {A}

${}^{A}\tilde{p} = {}^{A}T_{B}\,{}^{B}\tilde{p}$

same physical pt but two different frames

Rigid-Body Operator in Different Frames

- Expression of T in another frame (other than {O}):

$$\underset{\text{operation in }\{O\}}{T} \quad \leftrightarrow \quad \underset{\text{operation in }\{B\}}{T_B^{-1} T T_B}$$

T_B means ${}^{O}T_B$

Outline

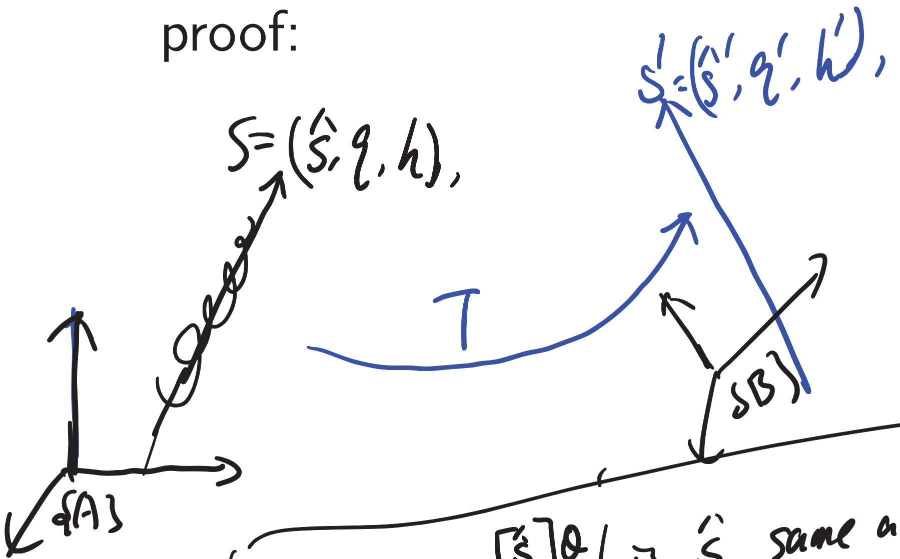
- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Rigid Operation on Screw Axis

- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

proof:



- Let's work an arbitrary frame {A} (rigidly attached to the screw axis)
- Let {B} be the frame obtained by apply T operation.

Assume $T = e^{[\hat{s}]\theta}$, is $\hat{s}$ same as S or S'?

Not at all

- the coordinate of S in {A} is the same as the coordinate of S' in {B} i.e. $\boxed{{}^A S = {}^B S'}$ — ①
- We also know

$${}^A T_B = T\,{}^A T_A \Rightarrow \boxed{T = {}^A T_B}$$

${}^B q' = {}^A q$

More Space

· Multiply ${}^{A}X_{B}$ to ①

$$\Rightarrow \quad {}^{A}X_{B}\, {}^{A}S = \underbrace{{}^{A}X_{B}\, {}^{B}S'} = {}^{A}S'$$

$${}^{A}T_{B}$$

$$T = [{}^{A}R_{B},\ {}^{A}p_{B}]$$

$$\Rightarrow \quad \boxed{{}^{A}S' = {}^{A}X_{B}\ {}^{A}S}$$

$$\Downarrow$$

$$T = (R, p)$$

$${}^{A}X_{B} = \begin{bmatrix} {}^{A}R_{B} & 0 \\ [{}^{A}p_{B}]\,{}^{A}R_{B} & {}^{A}R_{B} \end{bmatrix} \Longrightarrow [Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

$$\underset{6\times 1}{S'} = \underbrace{[Ad_T]}_{6\times 6}\ \underset{6\times 1}{S}$$