MEE5114 Advanced Control for Robotics

Lecture 5: Instantaneous Velocity of Moving Frames

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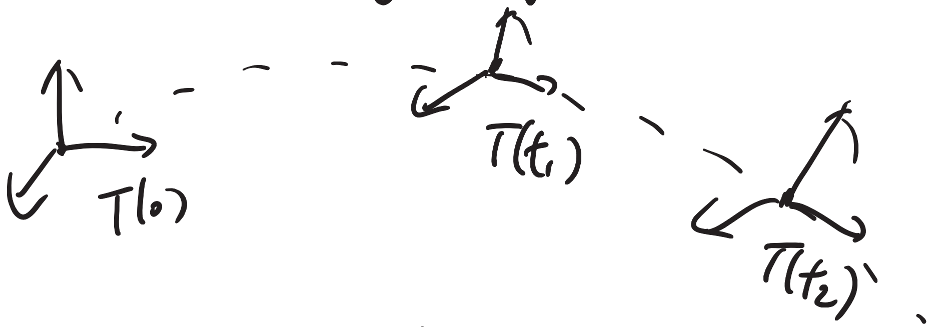
Outline

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

Given

- Frame trajectory. $T(t) = (R(t), p(t))$ (wrt $\{0\}$)



· Question: what is the velocity of frame at time t

twist / spatial velocity $\in \mathbb{R}^6$

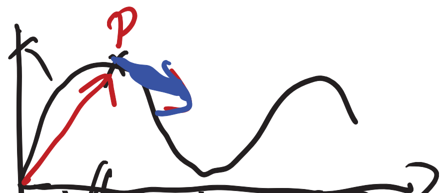
Outline

- $T(t) \in \mathbb{R}^{4\times 4}$: 4×4 matrix $\in SE(3)$
- $\log(T(t)) \longrightarrow \underset{\sim}{S}\theta$: is S the velocity of $T(t)$?

twist

no

e.g.:



$\mathbb{R}^2$

$SE(3)$

- Instantaneous Velocity of Rotating Frames

$p(t)$: "position" vector $\longleftrightarrow$ $S\theta \Leftrightarrow T(t)$: "position" coordinate

$\dot{p}(t)$: "velocity" vector

? what velocity ?

$\dot{T}(t)$ is velocity of $T(t)$?

4x4 matrix

How about $\log(\dot{T}(t))$?

✗

$\dot{T}(t) \notin SE(3)$

- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Rotating Frame (1/2)

- {A} frame is rotating with orientation $R_A(t)$ and velocity $\omega_A(t)$ at time t (Note: everything is wrt {O}-frame)

$\Downarrow$ orientation of A relative to {O} at time t

${}^{O}R_A(t)$

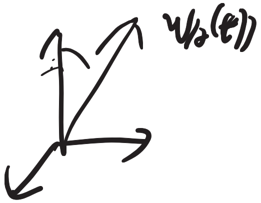
- Let $\hat{\omega}\theta = \log(R_A(t))$ be its exp. coordinate. : "position" coordinate
  - Note: $\hat{\omega}\theta$ means $R_A(t)$ can be obtained from the reference frame (say {O}-frame) by rotating about $\hat{\omega}$ by θ degree.
  - $\hat{\omega}\theta$ only describes the current orientation of {A} relative to {O}, it does not contain info about how the frame is rotating at time t.

Instantaneous Velocity of Rotating Frame (2/2)

- What is the relation between $\omega_A(t)$ and $R_A(t)$?

$$\frac{d}{dt}R_A(t) = [\omega_A(t)]R_A(t) \;\Rightarrow\; [\omega_A(t)] = \dot{R}_A(t)R_A^{-1}(t)$$

${}^oR_A(t) = [{}^o\hat{x}_A(t) \,\vdots\, {}^o\hat{y}_A(t) \,\vdots\, {}^o\hat{z}_A(t)]$ $\qquad \dot{R}_A = [\dot{\hat{x}}_A \;\; \dot{\hat{y}}_A \;\; \dot{\hat{z}}_A]$



$$\dot{\hat{x}}_A = \omega_A \times \hat{x}_A = [\omega_A]\hat{x}_A, \quad \dot{\hat{y}}_A = [\omega_A]\hat{y}_A, \quad \dot{\hat{z}}_A = [\omega_A]\hat{z}_A$$

$$\dot{R}_A = [\omega_A] R_A \Rightarrow [\omega_A] = \dot{R}_A R_A^{-1}$$

more precise: $\boxed{[{}^o\omega_A] = {}^o\dot{R}_A\,{}^oR_A^{-1}}$

what is ${}^A\omega_A$? $\qquad {}^A\omega_A = {}^AR_o\,{}^o\omega_A \Rightarrow [{}^A\omega_A] = [{}^AR_o\,{}^o\omega_A] = {}^AR_o[{}^o\omega_A]{}^AR_o^T$

(t)

$= {}^AR_o\,{}^o\dot{R}_A\,\cancel{{}^oR_A^{-1}\,{}^oR_A}$

$\Downarrow$

${}^A\omega_A$: velocity of A relative {o}, expressed in {A}

$= {}^oR_A^{-1}\,{}^o\dot{R}_A$

$[{}^A\omega_A] = {}^oR_A^{-1}\,{}^o\dot{R}_A$

Outline

- Instantaneous Velocity of Rotating Frames
- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Moving Frame (1/2)

- {A} moving frame with configuration $T_A(t)$ at time t undergoes a rigid body motion with velocity $\mathcal{V}_A(t) = (\omega, v)$ (Note: everything is wrt {O}-frame)

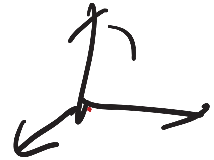
$^{O}T_A(t)$

$\hat{S}(t)\ \theta(t)$

- The exponential coordinate $\hat{S}\theta = \log(T_A(t))$ only indicates the current configuration of {A}, and does not tell us about how the frame is moving at time t.

Instantaneous Velocity of Moving Frame (2/2)

- What is the relation between $\mathcal{V}_A(t)$ and $T_A(t)$?



$$\frac{d}{dt}T_A(t) = [\mathcal{V}_A(t)]T_A(t) \Rightarrow [\mathcal{V}_A(t)] = \dot{T}_A(t)T_A^{-1}(t)$$

- a frame can be determined by direction vector of axes, and the origin

free vectors $\hat{x}_A$ $\hat{y}_A$ $\hat{z}_A$ $\qquad o_A$

$\Downarrow \qquad\qquad \Downarrow$

$\tilde{\hat{x}}_A = \begin{bmatrix} \hat{x}_A \\ 0 \end{bmatrix}$, $\quad \tilde{o}_A = \begin{bmatrix} o_A \\ 1 \end{bmatrix}$

$T_A = [\tilde{\hat{x}}_A \quad \tilde{\hat{y}}_A \quad \tilde{\hat{z}}_A \quad \tilde{o}_A]$

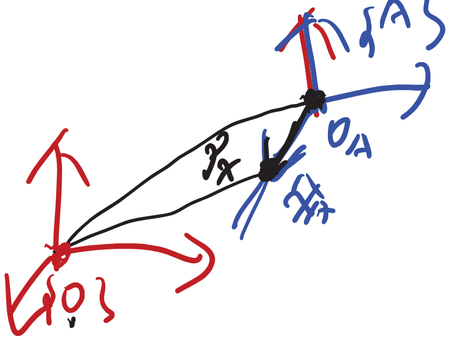
we want to show

$\dot{T}_A = [\dot{\tilde{\hat{x}}}_A \quad \cdots \quad] = [\mathcal{V}_A]\, T_A$

$\mathcal{V}_A = \begin{bmatrix} \omega \\ v_o \end{bmatrix}$

① $\dot{\tilde{\hat{x}}}_A = \begin{bmatrix} \dot{\hat{x}}_A \\ 0 \end{bmatrix}$, $\quad \dot{\hat{x}}_A = \underline{\omega} \times \hat{x}_A = [\omega]\hat{x}_A$

More Space



· prove it:

$$\hat{x}_A = \overrightarrow{O_A P_x} = P_x - O_A$$

$$\dot{\hat{x}}_A = \dot{P}_x - \dot{O}_A = \underline{\cancel{v_0} + \omega \times (\overrightarrow{OP_x})} - (\cancel{v_0} + \omega \times \overrightarrow{OO_A})$$

$$= \omega \times \hat{x}_A$$

$$\boxed{\dot{\hat{x}}_A = \omega \times \hat{x}_A} \Longrightarrow \dot{\tilde{\hat{x}}}_A = \begin{bmatrix} [\omega] & v_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_A \\ 0 \end{bmatrix}$$

$$\dot{\hat{y}}_A = \omega \times \hat{y}_A \quad \cdots \quad \dot{\hat{z}}_A = \omega \times \hat{z}_A$$

$$\cdot\ \dot{\tilde{O}}_A = \begin{bmatrix} \dot{O}_A \\ 0 \end{bmatrix} = \begin{bmatrix} v_0 + \underline{\omega \times} O_A \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} [\omega] & v_0 \\ 0 & 0 \end{bmatrix}}_{[\mathcal{V}_A]} \begin{bmatrix} O_A \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \dot{T}_A = [\mathcal{V}_A] T_A \quad \Rightarrow [\mathcal{V}_A] = \dot{T}_A T_A^{-1} \Rightarrow [{}^A\mathcal{V}_A] = T_A^{-1} \dot{T}_A$$