MEE5114 Advanced Control for Robotics

Lecture 6: Product of Exponential and Kinematics of Open Chain

Prof. Wei Zhang

CLEAR Lab
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China
https://www.wzhanglab.site/

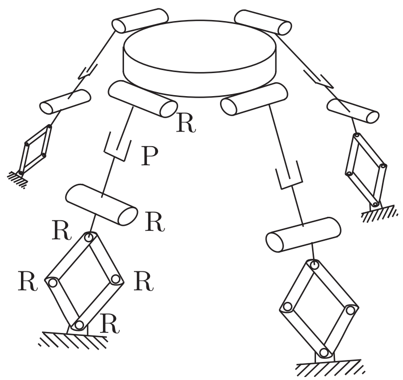
Outline

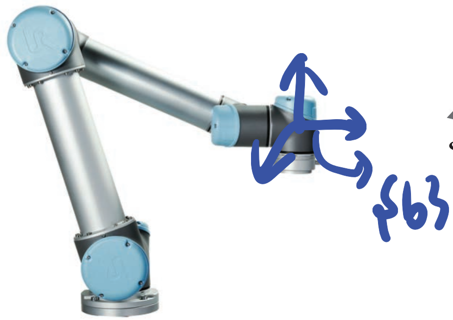
- Motivating Example
- Product of Exponential Formula Derivations
- Practice Example

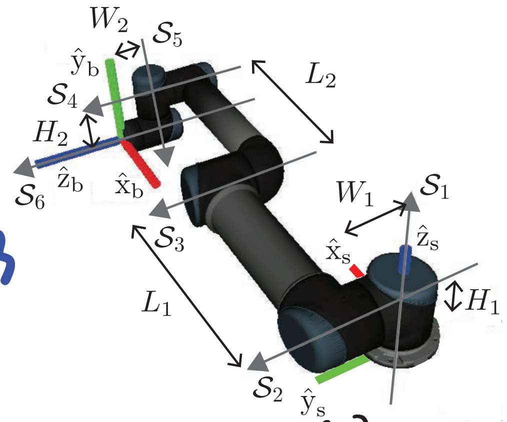
Kinematics

Robot: Multiple rigid bodies interconnected through joints

Kinematics is a branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the mass of each or the forces that caused the motion







pose = (R, p)
T

- **Forward Kinematics**: calculation of the configuration $T = (R, p)$ of the end-effector frame from joint variables $\theta = (\theta_1, \ldots, \theta_n)$

- **Velocity Kinematics (Next Lecture)**: Deriving the Jacobian matrix: linearized map from the joint velocities $\dot{\theta}$ to the spatial velocity $\mathcal{V}$ of the end-effector

Illustration Example (1/3)

Consider a 2R robot

- Three links and two joints θ_1, θ_2

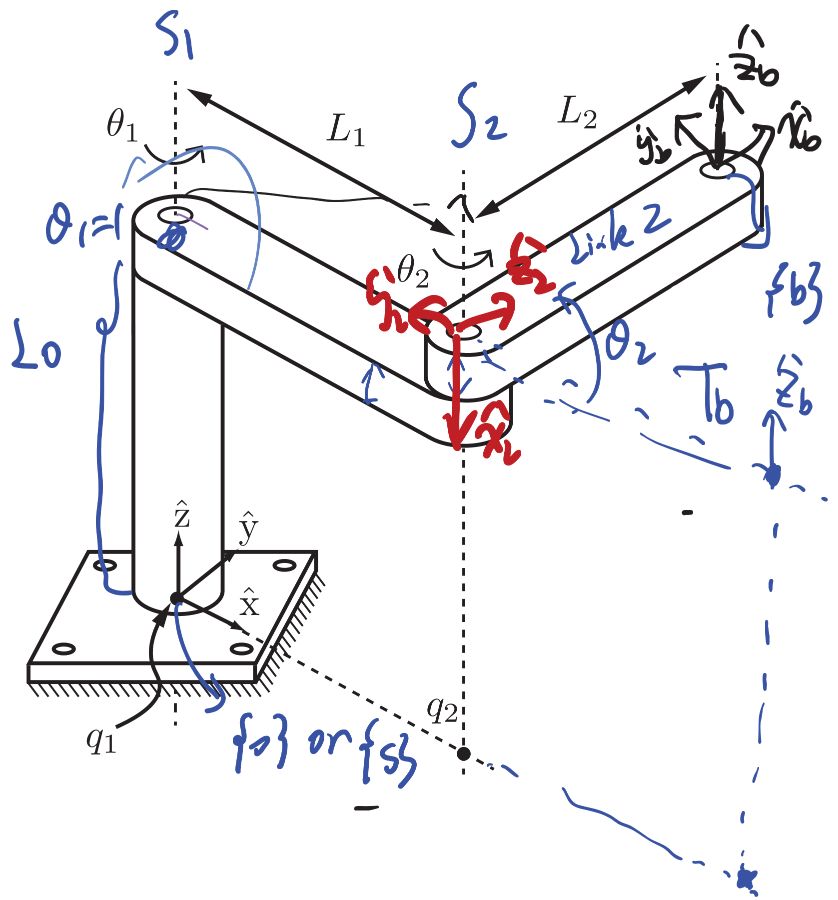
- Link/body frame attached to link i at joint i (one of possible choices)

- Fixed/world frame {s} frame , end-effector frame {b}

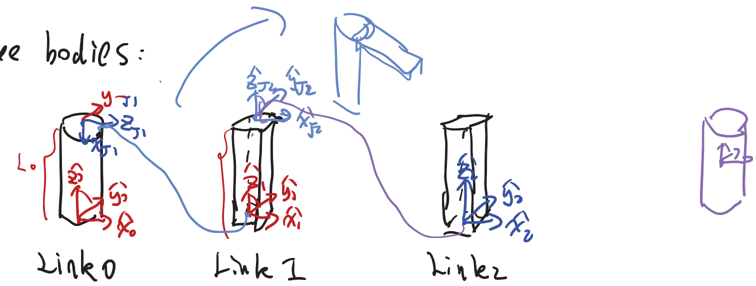
- **Goal**: compute ${}^sT_b(\theta_1, \theta_2)$: function of θ_1, θ_2

- Initial pose: $M \triangleq {}^sT_b(0,0)$

$$M = {}^sT_b(0,0) = \begin{bmatrix} 1 & 0 & 0 & L_1+L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Three bodies:



Define Joint frames:

Joint 1: $({}^0R_{J1}, {}^0p_{J1}) = (Rot(\hat{y}; \frac{\pi}{2}), (0, 0, L_0))$

Joint 2: $(I, (0, 0, L_1))$

Illustration Example (2/3)

- Fix joint 1 at $\theta_1 = 0$, rotate joint 2 by θ_2, we have ${}^sT_b(0, \theta_2)$

about

- Rigid body motion for link 2/{b}, represented by screw motion
- In coordinate-free way, $M \xrightarrow{\hat{T} = e^{[S_2]\theta_2}} \hat{T} \cdot M = e^{[S_2]\theta_2} M$

$\theta_1 = 0, \theta_2 = 0$

Remark

${}^2S_2$: constant

indep of θ

- In {s}/{0} frame ${}^sT_b(0, \theta_2) = e^{[{}^0S_2]\theta_2}\, T_b(0,0)$
  - ${}^0S_2$ is a func of θ_1, more precisely ${}^0S_2(\theta_1)$
  - Now $\theta_1 = 0$, define ${}^0\bar{S}_2 \triangleq {}^0S_2(0)$ ${}^0\bar{S}_2 = \begin{bmatrix} {}^0\omega_2 \\ {}^0v_2 \end{bmatrix}$

${}^0\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $h = 0$, ${}^0v_2 = -{}^0\omega_2 \times {}^0q_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$

$\Rightarrow {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix} \Rightarrow {}^sT_b(0, \theta_2) = e^{[{}^0\bar{S}_2]\theta_2} M$

Illustration Example (3/3)

- Fix joint 2 at θ_2, and rotate joint 1 by $\theta_1 \Rightarrow {}^sT_b(\theta_1, \theta_2)$

- rigid body motion $\Rightarrow {}^sT_b(\theta_1,\theta_2) = e^{[S_1]\theta_1}\, {}^sT_b(0, \theta_2)$

- ${}^0S_1$: indep of θ_1, θ_2 , ${}^0\bar{S}_1 = \begin{bmatrix} {}^0\omega_1 \\ {}^0v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^0v_1 = -{}^0\omega_1 \times {}^0q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad {}^sT_b(\theta_1,\theta_2) = \underbrace{e^{[{}^0\bar{S}_1]\theta_1}\, e^{[{}^0\bar{S}_2]\theta_2}}_{\text{product of exp}} M$$

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Notation Setup (1/2)

- Suppose that the robot has n joints and n links. Each joint has one degree of freedom represented by joint variable θ_i, $i = 1, \ldots, n$
  - θ_i: the joint angle (Revolute joint) or joint displacement (Primatic joint)

- Specify a fixed frame {s}: also referred to as frame {0}

- Attach frame $\{i\}$ to link i at joint i, for $i = 1, \ldots, n$

- Attach frame {b} at the end-effector: sometimes referred to as frame {n+1}

local coordinate of S_i

- ${}^i\mathcal{S}_i$: screw axis of joint i expressed in frame {i}

- ${}^0\mathcal{S}_i$: screw axis of joint i expressed in fixed frame {0} (i.e. frame {s})

${}^0S_i(\theta_1, \cdots, \theta_i)$

Notation Setup (2/2)

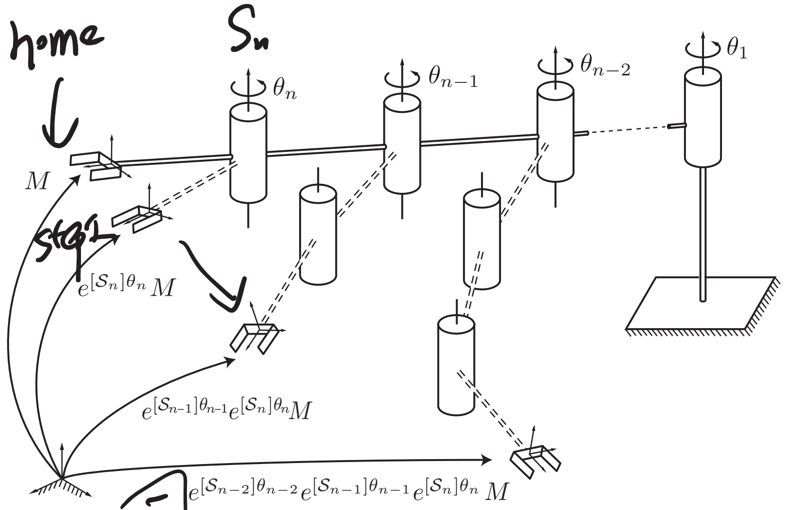
- For simplicity, we write configuration as T_{sb}, which is the same as ${}^sT_b$. Similarly, $T_{ij} = {}^iT_j$

- Note: ${}^i\mathcal{S}_i$ does not change when the robot moves (i.e. when θ changes), but ${}^0\mathcal{S}_i$ depends on $\theta_1, \ldots, \theta_i$. Sometimes, we write out the dependency explicitly, i.e. ${}^0\mathcal{S}_i(\theta_1, \ldots, \theta_i)$

- Define home position: $\theta_1 = 0, \ldots, \theta_n = 0$. This is the configuration when all the joint angles are zero. One can also choose other *fixed* angles as the home position

- Define ${}^0\bar{\mathcal{S}}_i = {}^0\mathcal{S}_i(0, \ldots, 0)$: the screw axis of joint i expressed in frame $\{0\}$, when the robot is at the home position.

Product of Exponential: Main Idea



- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$

step0 :

- Compute $M \triangleq T_{sb}(0, \ldots, 0)$: the configuration of end-effector when the robot is at home position $\theta_1 = \cdots = \theta_n = 0$

$S_n = {}^0\bar{S}_n$

step 1

- Apply screw motion to joint n: $T_{sb}(0, \ldots, 0, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$

this operator does not move S_{n-1}, S_{n-2} ...

- Apply screw motion to joint $n-1$ to obtain:

$$T_{sb}(0, \ldots, 0, \theta_{n-1}, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_{n-1}]\theta_{n-1}} e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

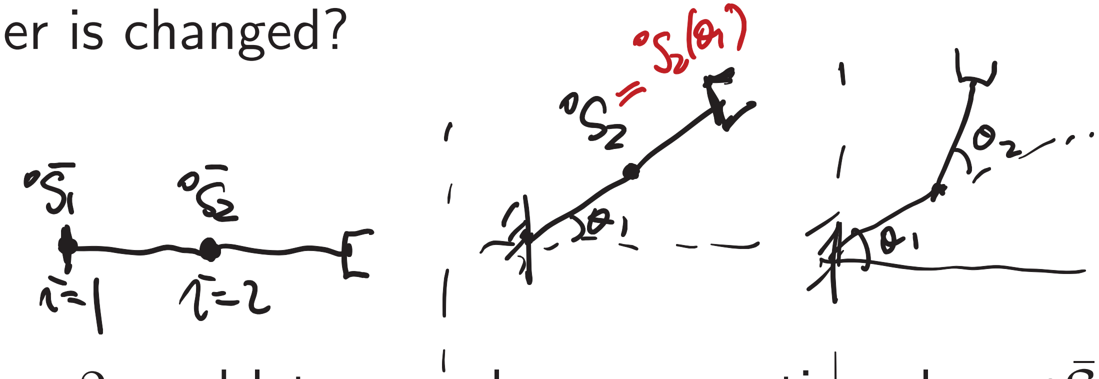
- After n screw motions, the overall forward kinematics:

$$T_{sb}(\theta_1, \ldots, \theta_n) = \boxed{\underbrace{e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} \cdots e^{[{}^0\bar{\mathcal{S}}_n]\theta_n}}_{\text{PoE}}} M$$

PoE: Screw Motions in Different Order (1/2)

- PoE was obtained by applying screw motions along screw axes ${}^0\bar{\mathcal{S}}_n$, ${}^0\bar{\mathcal{S}}_{n-1}$, ... What happens if the order is changed?

Top view of



- For simplicity, assume that $n = 2$, and let us apply screw motion along ${}^0\bar{\mathcal{S}}_1$ first:
  - $\boxed{T_{sb}(\theta_1, 0) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M}$
  - Now screw axis for joint 2 has been changed. The new axis ${}^0\mathcal{S}_2 = {}^0\mathcal{S}_2(\theta_1, 0) \neq {}^0\bar{\mathcal{S}}_2$.

$${}^sT_b(\theta_1, 0) = \boxed{e^{[{}^0\bar S_1]\theta_1} M}$$

$\neq {}^0\bar S_2$

$$\boxed{{}^sT_b(\theta_1, \theta_2) = \underbrace{e^{[{}^0S_2]\theta_2}}\ \underbrace{\boxed{{}^sT_b(\theta_1, 0)}}}$$ ✓

$${}^0\bar S_2 \xrightarrow{T = e^{[{}^0\bar S_1]\theta_1}} \underbrace{{}^0S_2(\theta_1)}_{6\times1} = \underbrace{[Ad_T]}_{6\times6}\ \underbrace{{}^0\bar S_2}_{6\times1}$$

where $T = e^{[{}^0\bar S_1]\theta_1}$

PoE: Screw Motions in Different Order (2/2)

- $T_{sb}(\theta_1, \theta_2) = e^{[{}^0\mathcal{S}_2]\theta_2} T_{sb}(\theta_1, 0)$

$S = \begin{bmatrix} w \\ v \end{bmatrix}$

① Recall: $[\underbrace{Rw}_{\omega}] = R[w]R^{-1}$

Fact: If $S' = [Ad_T] S \iff \underbrace{[S']}_{4\times 4} = \underset{4\times 4}{T} [S] T^{-1}$

$[S] = \begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix} \in se(3)$

Base on this Fact:

$$e^{[{}^0\bar{S}_2]\theta_2} = e^{[[Ad_T]{}^0\bar{S}_2]\theta_2} = e^{T[{}^0\bar{S}_2]T^{-1}\theta_2}$$

$$\overset{\checkmark}{=} T e^{[{}^0\bar{S}_2]\theta_2} T^{-1}$$

$$\Rightarrow {}^sT_b(\theta_1, \theta_2) = \underline{T e^{[{}^0\bar{S}_2]\theta_2} \cancel{T^{-1}}} \cancel{e^{[{}^0\bar{S}_1]\theta_1}} M = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} M$$

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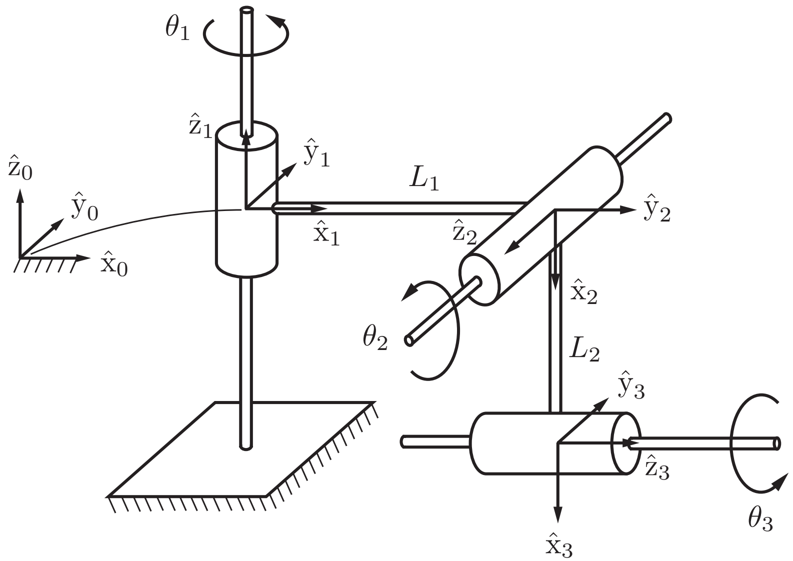
PoE Example: 3R Spatial Open Chain

- HW



More Discussions

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