MEE5114 Advanced Control for Robotics

Lecture 7: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

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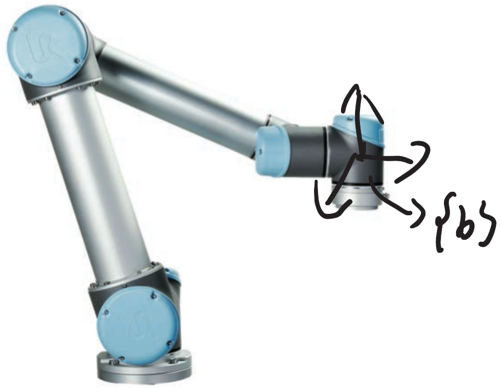
Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics



FK: Find the func of $T_b(\theta_1, \cdots, \theta_n)$

$\theta_1, \cdots, \theta_n \longmapsto T_b(\theta_1, \cdots, \theta_n)$ ${}^0\bar{S}_i$: meaning screw axis i when $\theta_i = ?$, express in {s}

Result: $T_b(\theta_1, \cdots, \theta_n) = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} \cdots e^{[{}^0\bar{S}_n]\theta_n} M$

- **Velocity Kinematics**: How does the velocity of {b} relate to the joint velocities $\dot{\theta}_1, \ldots, \dot{\theta}_n$; Note: {b}'s velocity is due to joint velocity.
- This depends on how to represent {b}'s velocity → Can we use $\dot{T}_b$ to represent velocity of {b} → 4x4
  - Twist representation → **Geometric Jacobian**

spatial

$\mathcal{V}_b = \begin{bmatrix} \omega \\ v \end{bmatrix}$, $\mathcal{V}_b(\theta, \dot{\theta})$: it turns out, $\mathcal{V}_b$ is a linear func of $\dot{\theta}$ $\Rightarrow$ $\mathcal{V}_b(\theta, \dot{\theta}) = \boxed{J(\theta)}\dot{\theta}$

this matrix is the Geometric

$J(\theta) \in \mathbb{R}^{6 \times n}$

  - Local coordinate of SE(3) → **Analytic Jacobian**

$\theta_1, \cdots, \theta_n \xrightarrow{FK} T_b(\theta_1, \cdots, \theta_n) = (R, p) \longrightarrow \underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}}_{x \in \mathbb{R}^6} \in \mathbb{R}^6$

p: (x, y, z)

R: (RPY: α, β, γ)

$g(\cdot)$

Outline

$x = g(\theta_1, \cdots, \theta_n) \Rightarrow \dot{x} = \left[\frac{\partial g}{\partial \theta}\right] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$

$\left[\frac{\partial g}{\partial \theta}\right]_{ij} = \frac{\partial g_i}{\partial \theta_j}$

6xn

Analytic Jacobian

- Background
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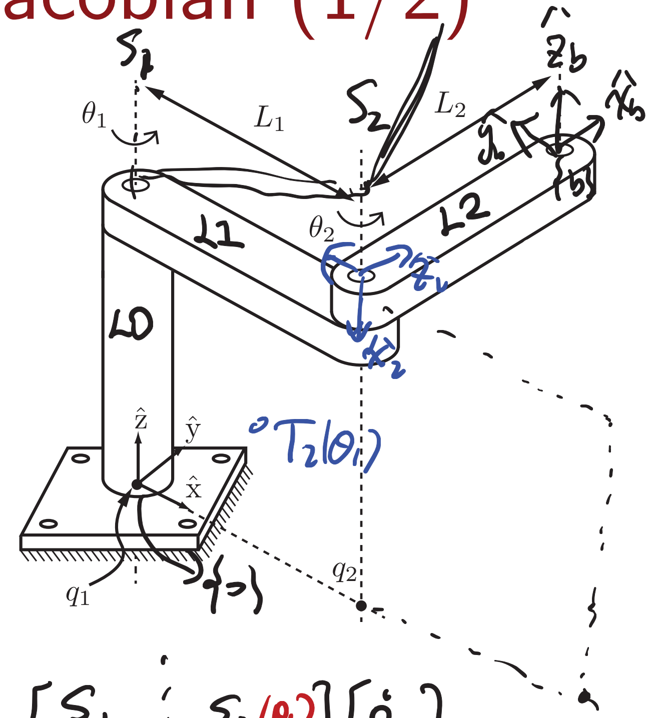
• For example: $p(\theta) = \begin{bmatrix} p_x(\theta) \\ p_y(\theta) \\ p_z(\theta) \end{bmatrix}$

$\dot{p}(\theta) = \left[\frac{\partial p}{\partial \theta}\right] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$

3xn

Simple Illustration Example: Geometric Jacobian (1/2)



- Coordinate-free

	Joint 1	Joint 2
Screw axis:	S_1	$S_2(\theta_1)$

indep of θ_1, θ_2

- Spatial velocity of each Link (when $\dot{\theta}_1, \dot{\theta}_2$)

Link 0: $\mathcal{V}_{L0} = 0 \in \mathbb{R}^6$; Link 1: $\mathcal{V}_{L1} = S_1\dot{\theta}_1$

Link 2: $\mathcal{V}_{L2} = \mathcal{V}_{L2/L1} + \mathcal{V}_{L1/L0} = S_2\dot{\theta}_2 + S_1\dot{\theta}_1 = [S_1 \; \vdots \; S_2(\theta_1)]\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$

$\Downarrow$

$\mathcal{V}_{L2/L0}$

$\{b\}$: $\mathcal{V}_b = \mathcal{V}_{L2} = [S_1 \; \vdots \; S_2(\theta_1)]\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix} = [J_1(\theta) \; \vdots \; J_2(\theta)]\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$

2nd column

1st-column of Geometric Jacobian

$J_i(\theta)$: the twist of $\{b\}$ when $\dot{\theta}_i = 1$, $\dot{\theta}_j = 0$, $j \neq i$

Simple Illustration Example: Geometric Jacobian (2/2)

- Computation: Let's work with $\{0\}$, $\underline{{}^0\mathcal{S}_1(\theta) = {}^0\mathcal{S}_1(\theta=0) = {}^0\bar{\mathcal{S}}_1}$ $= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0\mathcal{S}_2(\theta_1)$

Let $\theta_1 = 0$, ${}^0\bar{\mathcal{S}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$

$\underline{\theta_1 \neq 0}$,

$${}^0\bar{\mathcal{S}}_2 = {}^0\mathcal{S}_2(0) \xrightarrow{\hat{T}(\theta_1) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}} \underbrace{{}^0\mathcal{S}_2(\theta_1)}_{6\times 1} = \underbrace{\left[Ad_{\hat{T}(\theta_1)}\right]}_{6\times 6} \underbrace{{}^0\bar{\mathcal{S}}_2}_{6\times 1}$$

$$\Rightarrow {}^0J(\theta) = \left[{}^0\bar{\mathcal{S}}_1 \;\vdots\; \left[Ad_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2 \right]$$

Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

We have n joints

$$\mathcal{V} = J(\theta)\dot{\theta} = \underbrace{J_1(\theta)\dot{\theta}_1}_{J_i(\theta)} + \cdots + J_n(\theta)\dot{\theta}_n$$

$$= [J_1(\theta) \mid J_2(\theta) \mid \cdots \mid J_n(\theta)] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

Let $\dot{\theta}_i = 1$, $\dot{\theta}_j = 0$

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$\boxed{J_i(\theta) = \mathcal{S}_i(\theta)}$$

Geometric Jacobian: General Case (2/3)

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.

- The simplest way to write Jacobian is to use local coordinate:

$$ {}^iJ_i = {}^iS_i, \quad i = 1, \ldots, n $$

indep of θ

- In fixed frame $\{0\}$, we have

${}^0J = [{}^0J_1, {}^0J_2, \ldots, {}^0J_n]$

$$ {}^0J_i(\theta) = {}^0X_i(\theta)\, {}^iS_i, \quad i = 1, \ldots, n \tag{1} $$

- Recall: ${}^0X_i$ is the change of coordinate matrix for spatial velocities.

- Assume $\theta = (\theta_1, \ldots, \theta_n)$, then

$$ {}^0T_i(\theta) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^0X_i(\theta) = \left[\mathrm{Ad}_{{}^0T_i(\theta)}\right] \tag{2} $$

pose of frame $\{i\}$ relative to $\{0\}$

Geometric Jacobian: General Case (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i = 1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i = 2$, ${}^0S_2(\theta) = {}^0S_2(\theta_1) = \underbrace{\left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right]}\, {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$ $\theta_1 = 0$

$i=3$, ${}^0S_3(\theta) = {}^0S_3(\theta_1, \theta_2)$

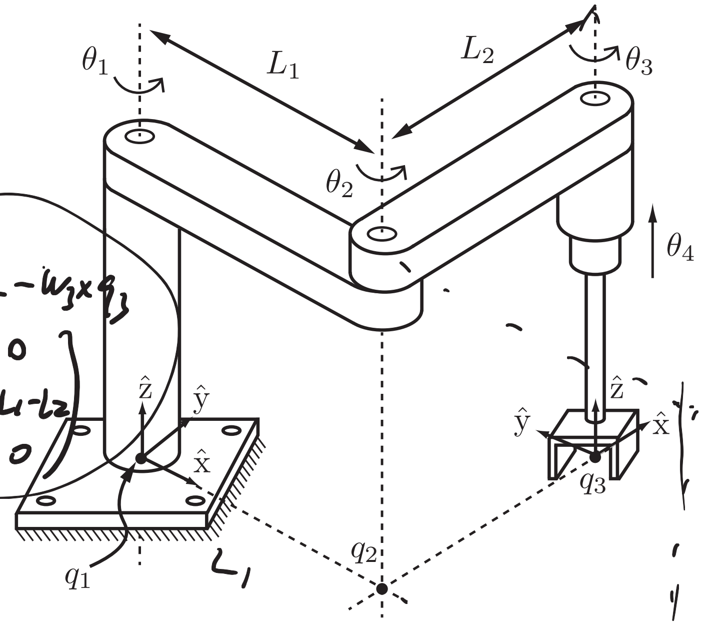
${}^0\bar{S}_3 = {}^0S_3(0,0)$ $\hat{T}_2(\theta_1,\theta_2) = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}$ $\underbrace{\left[\mathrm{Ad}_{\hat{T}(\theta_1,\theta_2)}\right]}_{6\times 6} {}^0\bar{S}_3$

  - For general i, we have

$$\boxed{\begin{aligned} {}^0J_i(\theta) &= {}^0S_i(\theta) = \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i \\ \text{where}\quad \hat{T}(\theta_1,\ldots,\theta_{i-1}) &\triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}} \end{aligned}} \tag{3}$$

Geometric Jacobian Example

- $J(\theta) = [\, S_1(\theta),\ S_2(\theta),\ S_3(\theta),\ S_4(\theta) \,]$



1°: Find screw axis at home position ($\theta_1 = \theta_2 \cdots = 0$)

$${}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}, \quad {}^0\bar{S}_3 = \begin{pmatrix} {}^0\omega_3 \\ {}^0v_3 \end{pmatrix}, \quad {}^0\omega_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad {}^0v_3 = -\omega_3 \times q_3 = \begin{pmatrix} 0 \\ -L_1-L_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1-L_2 \\ 0 \end{pmatrix}$$

L_1

$${}^0\bar{S}_4 = \begin{pmatrix} \omega_4 \\ v_4 \end{pmatrix}, \quad \text{pure linear motion}, \ h = \infty, \quad \omega_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

L_2

q_3

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{T}_2 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}$$

2°:

$${}^0J(\theta) = \left[\, {}^0\bar{S}_1,\ [Ad_{\hat{T}_1}]\,{}^0\bar{S}_2,\ [Ad_{\hat{T}_2}]\,{}^0\bar{S}_3,\ [Ad_{\hat{T}_3}]\,{}^0\bar{S}_4 \,\right]$$

$$\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1} \qquad \hat{T}_3 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_3]\theta_3}$$

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Analytic Jacobian

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be $\underbrace{\text{Cartesian}}_{\text{spherical}}$ + $\underbrace{\text{Euler angle}}_{\text{coordinate}}$ of end-effector frame → ZYZ ZYX
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame

$x = g(\theta)$ → $J_a(\theta) = \frac{\partial g}{\partial \theta}$

- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

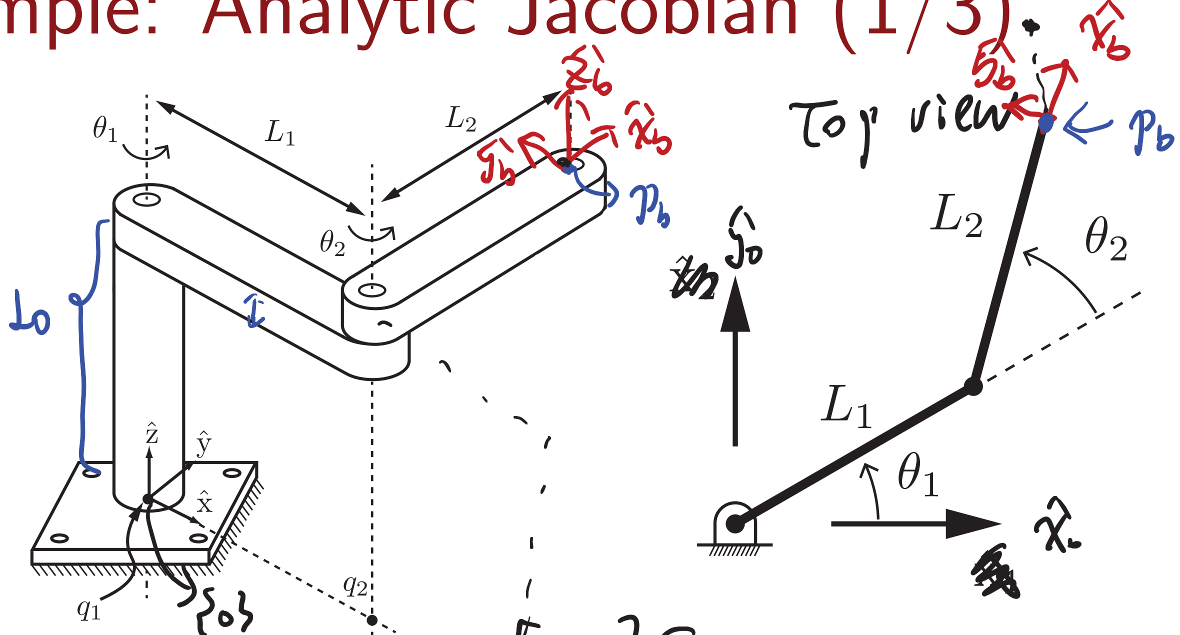
  - $E(x)$ can be ~~easily~~ found with given parameterization x

Simple Illustration Example: Analytic Jacobian (1/3)

task variable

$$\underbrace{{}^0\dot{p}_b}_{\text{task variable}} = \underbrace{\left[\frac{\partial g}{\partial \theta}\right]}_{\text{Analytic Jacobian}} \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$$

Analytic Jacobian $\rightarrow J_a(\theta)$



$${}^0p_b = \underbrace{\begin{cases} {}^0p_{b,x} = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) & \leftarrow g_1 \\ {}^0p_{b,y} = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) & \leftarrow g_2 \\ {}^0p_{b,z} = L_0 & \leftarrow g_3 \end{cases}}_{g(\theta_1,\theta_2)}$$

$$J_a(\theta) = \left[\frac{\partial g}{\partial \theta}\right] = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \\ \frac{\partial g_3}{\partial \theta_1} & \frac{\partial g_3}{\partial \theta_2} \end{bmatrix}$$

$J_a(\theta)$

$$= \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$$

Simple Illustration Example: Analytic Jacobian (2/3)

- Let ${}^0J(\theta)$ denote the Geometric Jacobian

$$ {}^0\mathcal{V}_b = {}^0J(\theta)\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}, \quad {}^0\mathcal{V}_b = \begin{bmatrix}{}^0\omega \\ {}^0v\end{bmatrix} $$

$$ {}^0\dot{p}_b = {}^0v + {}^0\omega \times {}^0p_b = -\underbrace{{}^0p_b \times}_{[{}^0p_b]}{}^0\omega + {}^0v = \left[-[{}^0p_b] \;\vdots\; I_{3\times3}\right]\begin{bmatrix}{}^0\omega \\ {}^0v\end{bmatrix} $$

$$ = \underbrace{\left[-[{}^0p_b(\theta)] \;\vdots\; I_{3\times3}\right]{}^0J(\theta)}_{J_a(\theta)}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} $$

Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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