MEE5114 Advanced Control for Robotics

Lecture 8: Rigid Body Dynamics

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Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Acceleration

- Given a rigid body with spatial velocity $\mathcal{V} = (\omega, v_o)$, its spatial acceleration is

$$\mathcal{A} = \dot{\mathcal{V}} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix}$$

  - Recall that: v_o is the velocity of the body-fixed particle coincident with frame origin o at the current time t.
  - Note: $\dot{\omega}$ is the angular acceleration of the body
  - $\dot{v}_o$ is not the acceleration of any body-fixed point!
  - In fact, $\dot{v}_o$ gives the rate of change in stream velocity of body-fixed particles passing through o

Spatial vs. Conventional Accel. (1/2)

- Why "$\dot{v}_o$ is not the acceleration of any body-fixed point"?

- Suppose $q(t)$ is the body fixed particle coincides with o at time t

- So by definition, we have $v_o(t) = \dot{q}(t)$, however, $\dot{v}_o(t) \neq \ddot{q}(t)$, where $\ddot{q}(t)$ is the conventional acceleration of the body-fixed point q

  - Note: $\dot{v}_o(t) = \lim_{\delta \to 0} \frac{v_o(t+\delta) - v_o(t)}{\delta}$

Spatial vs. Conventional Accel. (2/2)

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- If $q(t)$ is the body fixed particle coincides with o at time t, then we have

$$\ddot{q}(t) = \dot{v}_o(t) + \omega(t) \times \dot{q}(t)$$

Plücker Coordinate System and Basis Vectors (1/2)

- Recall coordinate-free concept: let $w \in \mathbb{R}^3$ be a free vector with {o} and {A} frame coordinate ${}^{o}w$ and ${}^{A}w$

Plücker Coordinate System and Basis Vectors (2/2)

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Work with Moving Reference Frame

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Derivative of Adjoint

- Suppose a frame {A}'s pose is $T_A = (R_A, p_A)$, and is moving at an instantaneous velocity $\mathcal{V}_A = (\omega, v)$. Then

$$\frac{d}{dt}\left([\mathrm{Ad}_{T_A}]\right) = \begin{bmatrix} [\omega] & [v] \\ [\omega] & 0 \end{bmatrix} [\mathrm{Ad}_{T_A}]$$

Spatial Cross Product

- Given two spatial velocities (twists) $\mathcal{V}_1$ and $\mathcal{V}_2$, their spatial cross product is:

$$\mathcal{V}_1 \times \mathcal{V}_2 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \times \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}$$

- Matrix representation: $\mathcal{V}_1 \times \mathcal{V}_2 = [\mathcal{V}_1 \times]\mathcal{V}_2$, where

$$[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}$$

- Roughly speaking, when a motion vector $\mathcal{V}$ is moving with a spatial velocity $\mathcal{Z}$ (e.g. it is attached to a moving frame) but is otherwise not changing, then

$$\dot{\mathcal{V}} = \mathcal{Z} \times \mathcal{V}$$

Spatial Cross Product: Properties (1/1)

- Assume A is moving wrt to O with velocity $\mathcal{V}_A$

$$^O\dot{X}_A = [^O\mathcal{V}_A \times]^O X_A$$

- $[X\mathcal{V}\times] = X[\mathcal{V}\times]X^T$, for any transformation X and twist $\mathcal{V}$

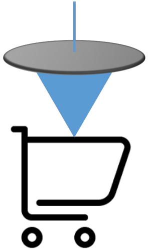
Spatial Acceleration with Moving Reference Frame

Consider a body with velocity $\mathcal{V}_{body}$ (wrt inertia frame), and ${}^O\mathcal{V}_{body}$ and ${}^B\mathcal{V}_{body}$ be its Plücker coordinates wrt {O} and {B}:

- ${}^B\mathcal{A}_{body} = \frac{d}{dt}\left({}^B\mathcal{V}_{body}\right) + {}^B\mathcal{V} \times {}^B\mathcal{V}_{body}$

- ${}^O\mathcal{A} = {}^O X_B \, {}^B\mathcal{A}$

Spatial Acceleration Example



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Spatial Force (Wrench)

- Consider a rigid body with many forces on it and fix an arbitrary point O in space
- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O
  - A moment n_O about point O
- **Spatial Force (Wrench):** is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

Spatial Force in Plücker Coordinate Systems

- Given a frame {A}, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

$$^{A}\mathcal{F} = \begin{bmatrix} ^{A}n_{o_A} \\ ^{A}f \end{bmatrix}$$

- Coordinate transform: $^{A}\mathcal{F} = {}^{A}X_{B}^{*}\,{}^{B}\mathcal{F}$ where $^{A}X_{B}^{*} = {}^{B}X_{A}{}^{T}$

Wrench-Twist Pair and Power

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

$$P = \left({}^A\mathcal{V}\right)^T \, {}^A\mathcal{F}$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by:$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$

- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \mathcal{V}^T \mathcal{F} = (\hat{\mathcal{S}}^T \mathcal{F})\dot{\theta} \triangleq \tau\dot{\theta}$$

- $\tau = \hat{\mathcal{S}}^T \mathcal{F} = \mathcal{F}^T \hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

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Rotational Inertia (1/2)

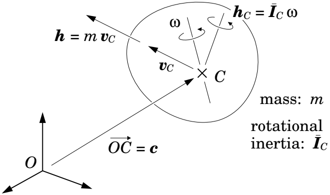
- Recall momentum for point mass:

Rotational Inertia (2/2)

- Rotational Inertia: $\bar{I} = \int_V \rho(r)[r][r]^T dr$
  - $\rho(\cdot)$ is the density function of the body
  - $\bar{I}$ depends on coordinate system
  - It is a constant matrix if the origin coincides with CoM

Spatial Momentum

- Consider a rigid body with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C
  - Linear momentum:
  - Angular momentum about CoM:
  - Angular momentum about a point O:
- Spatial Momentum:



Change Reference Frame for Momentum

- Spatial momentum transforms in the same way as spatial forces:

$$^{A}h = {}^{A}X_{C}^{*}\,{}^{C}h$$

Spatial Inertia

- Inertia of a rigid body defines linear relationship between velocity and momentum.

- Spacial inertia $\mathcal{I}$ is the one such that

$$h = \mathcal{I}\mathcal{V}$$

- Let {C} be a frame whose origin coincide with CoM. Then

$$^{C}\mathcal{I} = \begin{bmatrix} ^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

Spatial Inertia

- Spatial inertia wrt another frame {A}:

$$ {}^{A}\mathcal{I} = {}^{A}X_C^* \, {}^{C}\mathcal{I} \, {}^{C}X_A $$

- Special case: ${}^{A}R_C = I_3$

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Cross Product for Spatial Force and Momentum

Assume frame A is moving with velocity ${}^{A}\mathcal{V}_{A}$

- ${}^{A}\left[\frac{d}{dt}\mathcal{F}\right] = \frac{d}{dt}\left({}^{A}\mathcal{F}\right) + {}^{A}\mathcal{V} \times^{*} {}^{A}\mathcal{F}$

- ${}^{A}\left[\frac{d}{dt}h\right] = \frac{d}{dt}\left({}^{A}h\right) + {}^{A}\mathcal{V} \times^{*} {}^{A}h$

Newton-Euler Equation

- Newton-Euler equation:

$$\mathcal{F} = \frac{d}{dt}h = \mathcal{I}\mathcal{A} + \mathcal{V} \times^* \mathcal{I}\mathcal{V}$$

- Adopting spatial vectors, the Newton-Euler equation has the same form in any frame

Derivations of Newton-Euler Equation

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More Discussions

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