MEE5114 Advanced Control for Robotics

Lecture 8: Rigid Body Dynamics

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Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Acceleration

- Given a rigid body with spatial velocity $\mathcal{V} = (\omega, v_o)$, its spatial acceleration is

coordinate-free

$$\mathcal{A} = \dot{\mathcal{V}} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix}, \quad \mathcal{A} = \lim_{\delta \to 0} \frac{\mathcal{V}(t+\delta) - \mathcal{V}(t)}{\delta}$$

  - Recall that: v_o is the velocity of the body-fixed particle coincident with frame origin o at the current time t.

  - Note: $\dot{\omega}$ is the angular acceleration of the body

    $v_o = \dot{q}(t)$, for some body-fixed pt

  - $\dot{v}_o$ is not the acceleration of any body-fixed point! but $\dot{v}_o \neq \ddot{q}(t)$

  - In fact, $\dot{v}_o$ gives the rate of change in stream velocity of body-fixed particles passing through o

Spatial vs. Conventional Accel. (1/2)

- Why "$\dot{v}_o$ is not the acceleration of any body-fixed point"?

- Suppose $q(t_0)$ is the body fixed particle coincides with o at time t_0

  $q(t_0)$ "=" o "

- So by definition, we have $v_o(t_0) = \dot{q}(t_0)$, however, $\dot{v}_o(t_0) \neq \ddot{q}(t_0)$, where $\ddot{q}(t)$ is the conventional acceleration of the body-fixed point q

  $q_1(t_0+\delta) \neq q(t_0+\delta)$ $\quad q(t_0)$

  - Note: $\dot{v}_o(t_0) \triangleq \lim_{\delta\to 0} \frac{v_o(t_0+\delta)-v_o(t_0)}{\delta}$ $\quad \neq \dot{q}(t_0)$

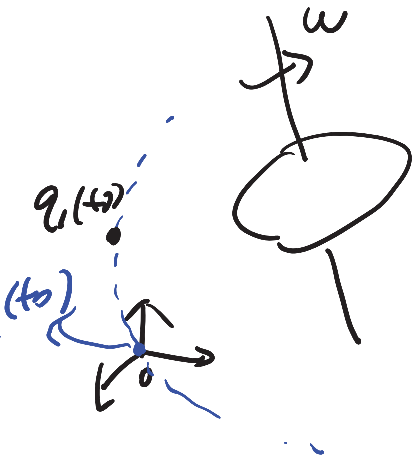
At time $t=t_0$, $\quad q(t_0)=o$, $\quad$, $\quad V_o(t_0)=\dot{q}(t_0)$

At time t_0

time $t=t_0+\delta$, $\quad q_1(t_0+\delta)=o$, $\quad V_o(t_0+\delta) \neq \dot{q}(t_0+\delta)$

another body-fixed particle

$= \dot{q}_1(t_0+\delta)$



Note: q_1 and q are different points.

$\dot{q}_1(t_0+\delta) \neq \dot{q}(t_0+\delta)$

Spatial vs. Conventional Accel. (2/2)

$$\frac{v_o(t_0+\delta) - v_o(t_0)}{\delta} \neq \frac{\dot{q}(t_0+\delta) - \dot{q}(t_0)}{\delta} \xrightarrow{\delta \to 0} \ddot{q}(t_0)$$

By definition:

$$\boxed{\dot{q}(t) = v_o(t) + \omega(t) \times q(t)} \leftarrow \text{holds for all } t$$

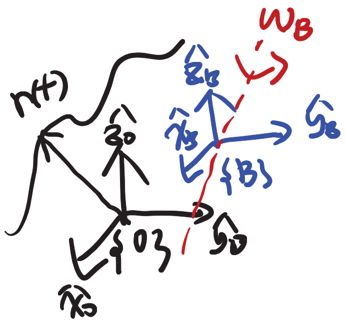
$$\boxed{\ddot{q}(t) = \dot{v}_o(t) + \dot{\omega}(t) \times q(t) + \omega(t) \times \dot{q}(t)} \leftarrow \checkmark$$

- At $t = t_0$, if $q(t_0) = 0$ $\quad \boxed{\ddot{q}(t_0) = \dot{v}_o(t_0) + \omega(t_0) \times \dot{q}(t_0)}$
- If $q(t)$ is the body fixed particle coincides with o at time t, then we have

$$\boxed{\ddot{q}(t) = \dot{v}_o(t) + \omega(t) \times \dot{q}(t)}$$

Plücker Coordinate System and Basis Vectors (1/2)

- Recall coordinate-free concept: let ~~v~~ $r \in \mathbb{R}^3$ be a free vector with $\{o\}$ and $\{$~~A~~ $B\}$ frame coordinate ${}^o r$ and ${}^B r$



$${}^o r = \begin{bmatrix} {}^o r_x \\ {}^o r_y \\ {}^o r_z \end{bmatrix} \in \mathbb{R}^3 \iff r = [\hat{x}_o \ \ \hat{y}_o \ \ \hat{z}_o]\, {}^o r \quad \cdots\cdots ①$$

$${}^B r = \begin{bmatrix} {}^B r_x \\ {}^B r_y \\ {}^B r_z \end{bmatrix} \in \mathbb{R}^3 \iff r = [\hat{x}_B \ \ \hat{y}_B \ \ \hat{z}_B]\, {}^B r \quad --- ②$$

$\Downarrow$ express this "physics" using $\{o\}$

$$\Rightarrow {}^o r = \underbrace{[{}^o\hat{x}_B \ \ {}^o\hat{y}_B \ \ {}^o\hat{z}_B]}_{{}^oR_B} {}^B r$$

${}^oR_B \leftarrow$ change of coord

- ① $\Rightarrow$ $$\boxed{\dot{r} = [\hat{x}_o \ \ \hat{y}_o \ \ \hat{z}_o]\frac{d}{dt}({}^o r)}$$

$\Downarrow$ apparent derivative $\triangleq {}^o\dot{r}$

use $\{o\}$-frame to express:

$${}^o(\dot{r}) = \underbrace{[{}^o\hat{x}_o \ \ {}^o\hat{y}_o \ \ {}^o\hat{z}_o]}_{I_{3\times 3}} \frac{d}{dt}({}^o r)$$

$${}^o(\dot{r}) = \frac{d}{dt}({}^o r)$$

Plücker Coordinate System and Basis Vectors (2/2)

- ② $\Rightarrow \quad r = [\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]\, {}^B r \qquad \dot{\hat{x}}_B = \omega_B \times \hat{x}_B$

$$\underbrace{\left(\dot{r} \overset{?}{=} \frac{d}{dt}({}^B r) \; \times \right)}$$

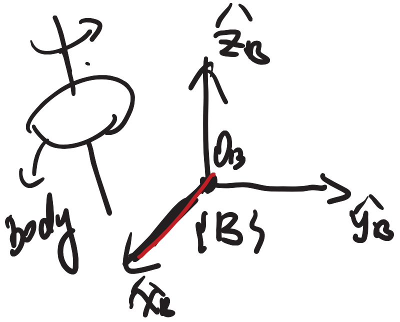
$$\Rightarrow \quad \dot{r} = \underbrace{[\dot{\hat{x}}_B \;\; \dot{\hat{y}}_B \;\; \dot{\hat{z}}_B]}\, {}^B r + \underbrace{[\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]}\, \frac{d}{dt}({}^B r)$$

$$= \omega_B \times \underbrace{[\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]}\, {}^B r + [\quad]\, \frac{d}{dt}({}^B r) \;\; \ldots$$

use $\{B\}$ frame to express the above equ.

$$\boxed{{}^B(\dot{r})} = \underbrace{{}^B\omega_B \times {}^B r}_{\text{accounts for coordinate frame axes is moving}} + \underbrace{\boxed{\frac{d}{dt}({}^B r)}}_{\text{due to changes in coordinate}} \rightarrow {}^B \mathring{r}$$

$${}^B\left(\frac{d}{dt}(r)\right) \qquad {}^B\dot{r}$$



- $${}^B\mathcal{V}_{body} = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}$$, means rotating about $\hat{x}_B$ at unit speed

  $\iff \mathcal{V}_{body} = e_{B_1}$ motion basis vector

  $$= \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}$$, ... $\hat{y}_B$ . . . . . .

  $\mathcal{V}_{body} = e_{B_2}$

  ⋮

  $$\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}$$, linear motion along $\hat{x}_B$ at unit speed

Given $\{B\}$ frame

$\{e_{B_1}, e_{B_2} \cdots e_{B_6}\}$ - 6 dim

motion basis vectors

coordinate free

$$\mathcal{V}_{body} = \alpha_1 e_{B_1} + \alpha_2 e_{B_2} + \cdots + \alpha_6 e_{B_6}, \quad \text{where} \begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_6\end{bmatrix} = {}^B\mathcal{V}_{body}$$

use $\{o\}$ frame to represent above "physics"

$$\boxed{{}^o\mathcal{V}_{body} = \alpha_1\, {}^o e_{B_1} + \alpha_2\, {}^o e_{B_2} + \cdots + \alpha_6\, {}^o e_{B_6} = [{}^o e_{B_1} \;\; {}^o e_{B_2} \;\cdots\; {}^o e_{B_6}]\, {}^B\mathcal{V}_{body}}$$

- ${}^o e_{B_i}$: can be computed frome "physics" and twist def.

  unit speed rotation about $\overrightarrow{o_B x_B}$ expressed in $\{o\}$

- $\underbrace{[{}^0e_{B_1} \quad {}^0e_{B_2} \quad \cdots \quad {}^0e_{B_6}]}_{6\times 6} = \underbrace{{}^0X_B}_{6\times 6} = [Ad_{{}^0T_B}]$

- $A_{body} = \frac{d}{dt}(V_{body})$; $V_{body} = [e_{B_1} \quad \cdots \quad e_{B_6}]\, {}^B\mathcal{V}_{body}$

$\Rightarrow A_{body} = \frac{d}{dt}(\mathcal{V}_{body}) = \underbrace{[\dot{e}_{B_1}, \dot{e}_{B_2} \quad \cdots \quad \dot{e}_{B_6}]}\, {}^B\mathcal{V}_{body} + [e_{B_1} \quad \cdots \quad e_{B_6}]\, \underbrace{\frac{d}{dt}({}^B\mathcal{V}_{body})}_{{}^B\mathring{\mathcal{V}}_{body}}$

If $\{B\}$ - does not change

$\dot{\mathcal{V}}_{body} = \underbrace{[e_{B_1} \quad \cdots \quad e_{B_6}]}\, {}^B\mathring{\mathcal{V}}_{body} \Leftarrow$ express this in $\{B\}$ $\underbrace{{}^B(\dot{\mathcal{V}}_{body})}_{{}^BA_{body}} = {}^B\mathring{\mathcal{V}}_{body} = \frac{d}{dt}({}^B\mathcal{V}_{body})$

${}^BA_{body} =$

special case: ${}^0A = {}^0\mathring{\mathcal{V}}_{body}$

If $\{B\}$ - changes over time, ${}^BA \neq {}^B\mathring{\mathcal{V}}_{body}$

- Then the key is to compute $[\dot{e}_{B_1} \quad \dot{e}_{B_2} \quad \cdots \quad \dot{e}_{B_6}] \Leftarrow$ can be computed purely

Work with Moving Reference Frame

by physics (see featherstone)

- Now let's work with {0} frame to find the derivative

$\Rightarrow$ we need to compute: $[\,{}^{0}\dot{e}_{B_1} \;\; {}^{0}\dot{e}_{B_2} \;\cdots\; {}^{0}\dot{e}_{B_6}] = {}^{0}\dot{X}_B = \frac{d}{dt}[Ad_{{}^{0}T_B}]$

let's denote: ${}^{0}T_B = (R, p) \Rightarrow \frac{d}{dt}\left(\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}\right) = \begin{bmatrix} \dot{R} & 0 \\ ([p]R)' & \dot{R} \end{bmatrix}$

{B}-frame has instantaneous velocity $\mathcal{V}_B = \begin{bmatrix} \omega \\ v \end{bmatrix}$

Note: $\dot{R} = \omega \times R$, $\dot{p} = v + \omega \times p$, $[R\omega] = R[\omega]R^T$

$[\omega_1 \times \omega_2] = [\omega_1][\omega_2] - [\omega_2][\omega_1]$ ~ Jacobi's identity

$\Rightarrow$ After some computation, $\frac{d}{dt}(Ad_{T_B}) = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}(Ad_{T_B})$

$$\underbrace{{}^{0}\dot{X}_B}_{6\times 6} = \underbrace{\begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}}_{6\times 6} \underbrace{{}^{0}X_B}_{6\times 6}$$

Define: $\begin{bmatrix} [w] & 0 \\ [v] & [w] \end{bmatrix} \triangleq [\mathcal{V}_B \times]$, $\boxed{{}^0\dot{X}_B = \mathcal{V}_B \times {}^0X_B}$

$\boxed{{}^0\dot{R}_B = \omega_B \times {}^0R_B}$

In coordinate free: $\dot{e}_{B_1} = \mathcal{V}_B \times e_{B_1}$, $\dot{e}_{B_2} = \mathcal{V}_B \times e_{B_2}$ . . . .

Derivative of Adjoint

- Suppose a frame $\{A\}$'s pose is $T_A = (R_A, p_A)$, and is moving at an instantaneous velocity $\mathcal{V}_A = (\omega, v)$. Then

$$\frac{d}{dt}([\mathrm{Ad}_{T_A}]) = \begin{bmatrix} [\omega] & \overset{0}{\cancel{[v]}} \\ [v] & [\omega] \end{bmatrix} [\mathrm{Ad}_{T_A}]$$

$${}^0\dot{X}_A = \mathcal{V}_A \times {}^0X_A$$

$$\dot{X}_A = \mathcal{V}_A \times X_A$$

$${}^A\dot{X}_A = {}^A\mathcal{V}_A \times {}^AX_A$$

Spatial Cross Product

- Given two spatial velocities (twists) $\mathcal{V}_1$ and $\mathcal{V}_2$, their spatial cross product is:

$$\underbrace{\mathcal{V}_1 \times \mathcal{V}_2}_{\text{Lie Bracket}} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \times \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}$$

- Matrix representation: $\mathcal{V}_1 \times \mathcal{V}_2 = [\mathcal{V}_1 \times]\mathcal{V}_2$, where

$$[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}$$

- Roughly speaking, when a motion vector $\mathcal{V}$ is moving with a spatial velocity $\mathcal{Z}$ (e.g. it is attached to a moving frame) but is otherwise not changing, then

$$\dot{\mathcal{V}} = \mathcal{Z} \times \mathcal{V}$$

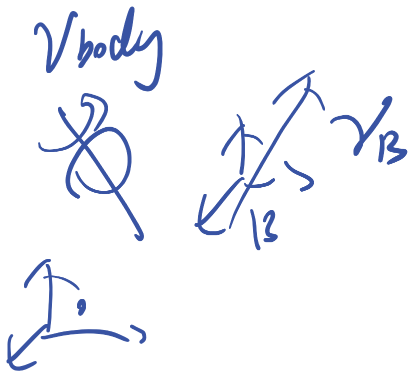
Spatial Cross Product: Properties (1/1)

- Assume A is moving wrt to O with velocity $\mathcal{V}_A$

$$^{O}\dot{X}_A = [^{O}\mathcal{V}_A \times]^{O}X_A$$

- $[X\mathcal{V}\times] = X[\mathcal{V}\times]X^T$, for any transformation X and twist $\mathcal{V}$

$$[R\omega] = R[\omega]R^T$$



Spatial Acceleration with Moving Reference Frame

Consider a body with velocity $\mathcal{V}_{body}$ (wrt inertia frame), and ${}^O\mathcal{V}_{body}$ and ${}^B\mathcal{V}_{body}$ be its Plücker coordinates wrt {O} and {B}:

- ${}^B\mathcal{A}_{body} = \underbrace{\frac{d}{dt}\left({}^B\mathcal{V}_{body}\right)}_{\text{apparent derivative}} + \underbrace{{}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body}}_{\text{due to frame \{B\} is moving}}$

$${}^B\left(\frac{d}{dt}(\mathcal{V}_{body})\right) = {}^B\dot{\mathcal{V}}_{body}$$

- ${}^O\mathcal{A} = {}^O X_B \, {}^B\mathcal{A}$

$${}^O\mathcal{A}_{body} = \frac{d}{dt}\left({}^O\mathcal{V}_{body}\right) = \frac{d}{dt}\left({}^O X_B \, {}^B\mathcal{V}_{body}\right) = \underbrace{{}^O\dot{X}_B \, {}^B\mathcal{V}_{body}} + {}^O X_B \, {}^B\dot{\mathcal{V}}_{body}$$

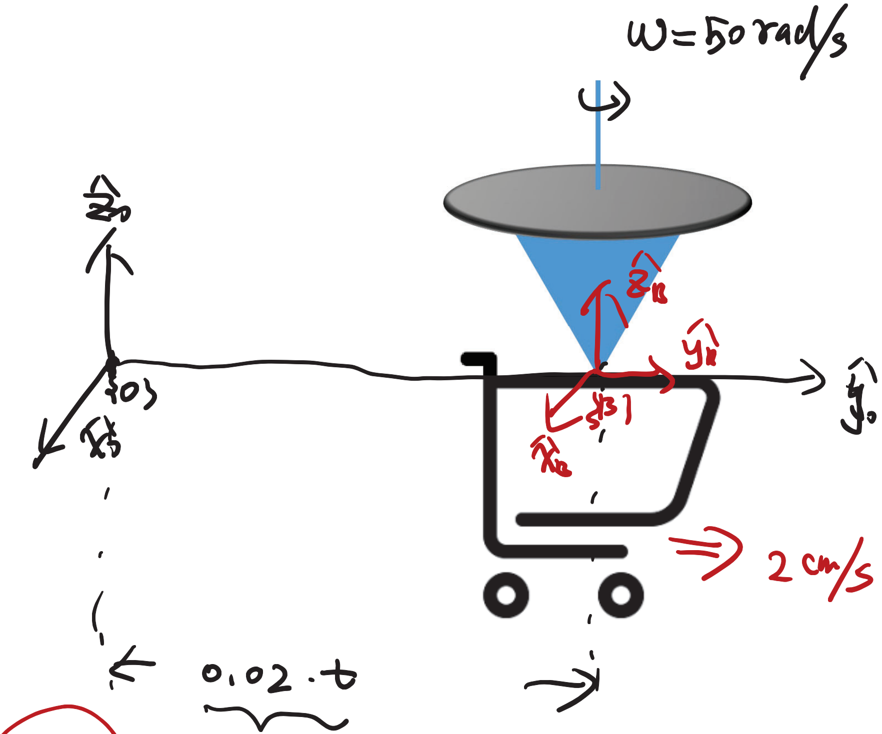
$$= \left[{}^O\mathcal{V}_B \times\right] {}^O X_B \, {}^B\mathcal{V}_{body} + {}^O X_B \, {}^B\dot{\mathcal{V}}_{body}$$

$$= {}^O X_B \left\{ \underbrace{{}^B X_O \left[{}^O\mathcal{V}_B \times\right] {}^O X_B} \, {}^B\mathcal{V}_{body} + {}^B\dot{\mathcal{V}}_{body} \right\} = {}^O X_B \left\{ \underbrace{{}^B\dot{\mathcal{V}}_{body} + {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body}}_{{}^B\mathcal{A}_{body}} \right\}$$

$$\left[{}^B X_O \, {}^O\mathcal{V}_B \times\right] = \left[{}^B\mathcal{V}_B \times\right]$$

Spatial Acceleration Example

Find ${}^B A_{top}$



Method 1: ${}^B A_{top} = {}^B X_0 \, {}^0 A_{top} = {}^B X_0 \frac{d}{dt}\left({}^0 \mathcal{V}_{top}\right)$

$${}^0 \mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ \hline 50\cdot(0.02t) \\ 0.02 \\ 0 \end{bmatrix}, \qquad {}^0 A_{top} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$${}^B A_{top} = {}^B X_0 \, {}^0 A_{top} = \begin{bmatrix} I & \\ [{}^B p_0] & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Method 2:

$${}^B \mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ \hline 0 \\ 0.02 \\ 0 \end{bmatrix}$$

$${}^B A_{top} = \underline{{}^B \dot{\mathcal{V}}_{top}} + {}^B \mathcal{V}_{B\,frame} \times {}^B \mathcal{V}_{top}$$

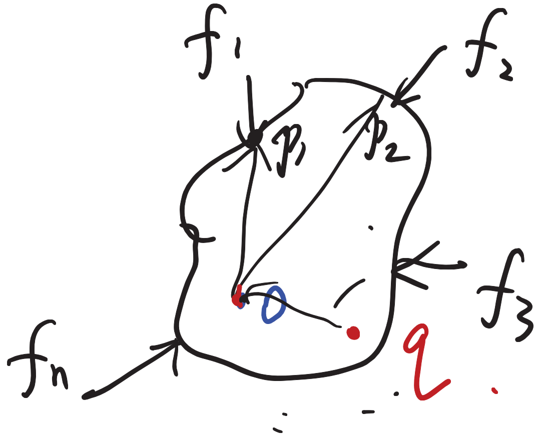
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0.02 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Force (Wrench)

- Consider a rigid body with many forces on it and fix an arbitrary point O in space



- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O

    $$f = \sum f_i$$

  - A moment n_O about point O

    $n_O \in \mathbb{R}^3$ $$n_O = \sum_i (\overrightarrow{Op_i}) \times f_i$$

- **Spatial Force (Wrench):** is given by the 6D vector

$$V_q = V_o + \omega \times \overrightarrow{oq}$$

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

what if we change reference point to q

$$n_q = \sum_i (\overrightarrow{qp_i}) \times f_i = n_O + \sum_i (\overrightarrow{qp_i} - \overrightarrow{Op_i}) \times f_i$$

$$= n_O + \overrightarrow{qO} \times \sum_i f_i = n_O + \overrightarrow{qO} \times f$$

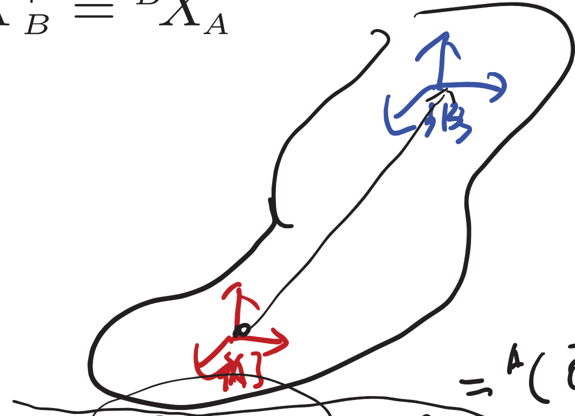
Spatial Force in Plücker Coordinate Systems $= n_O + f \times \vec{OQ}$

- Given a frame $\{A\}$, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

$${}^B\mathcal{F} = \begin{bmatrix} {}^B n_{O_B} \\ {}^B f \end{bmatrix} \qquad {}^A\mathcal{F} = \begin{bmatrix} {}^A n_{O_A} \\ {}^A f \end{bmatrix} \qquad {}^A T_B = ({}^A R_B, {}^A p_B)$$

- Coordinate transform: ${}^A\mathcal{F} = {}^A X_B^* \, {}^B\mathcal{F}$ where ${}^A X_B^* = {}^B X_A{}^T$

$$\boxed{{}^A f = {}^A R_B \, {}^B f} \quad \text{— ①}$$



moment: coord-free : $n_{O_A} = n_{O_B} + \vec{O_A O_B} \times f$

choose {A}-free to express : ${}^A n_{O_A} = {}^A n_{O_B} + {}^A(\vec{O_A O_B}) \times {}^A f$

$$= {}^A(\vec{O_A O_B} \times f) = {}^A R_B \, {}^B(\vec{O_A O_B} \times f)$$

$$= {}^A R_B \, {}^B n_{O_B} + {}^A R_B(-{}^B p_A \times {}^B f)$$

$$= {}^A R_B \, {}^B n_{O_B} - {}^A R_B [{}^B p_A] \, {}^B f \quad \text{— ②}$$

$$\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Rightarrow \begin{bmatrix} {}^A n_{O_A} \\ {}^A f \end{bmatrix} = \begin{bmatrix} {}^A R_B & -{}^A R_B [{}^B p_A] \\ 0 & {}^A R_B \end{bmatrix} \begin{bmatrix} {}^B n_{O_B} \\ {}^B f \end{bmatrix}$$

$$\Rightarrow \quad \triangleq \underline{{}^{A}X_{B}^{*}}$$

$$\boxed{{}^{A}X_{B}^{*} = \left({}^{B}X_{A}\right)^{T}}$$

6×6

$${}^{B}X_{A} = \begin{bmatrix} {}^{B}R_{A} & 0 \\ [{}^{B}p_{A}]\,{}^{B}R_{A} & {}^{B}R_{A} \end{bmatrix}$$

Wrench-Twist Pair and Power

$p = f^T v = \langle f, v \rangle$

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

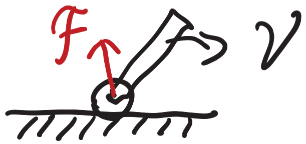
scalar

$$P = \underbrace{({}^A\mathcal{V})^T}_{6\times 1} \underbrace{{}^A\mathcal{F}}_{6\times 1} = {}^A\mathcal{F}^T \, {}^A\mathcal{V}$$

$$= \underbrace{({}^A\omega)^T \, {}^A n_{o_A}}_{\text{rotational power}} + \underbrace{{}^A v_{o_A}^T \, {}^A f}$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by: $\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$



- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \mathcal{V}^T\mathcal{F} = \boxed{(\hat{\mathcal{S}}^T\mathcal{F})\dot{\theta}} \triangleq \tau\dot{\theta}$$

$$(\hat{\mathcal{S}}\dot{\theta})^T\mathcal{F}$$

$\triangleq \tau$ scalar

- $\tau = \hat{\mathcal{S}}^T\mathcal{F} = \mathcal{F}^T\hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

q

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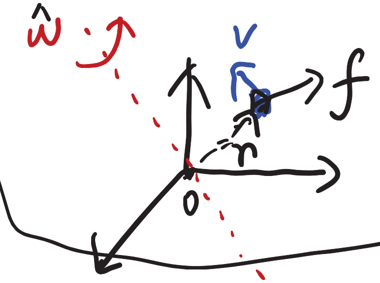
Rotational Inertia (1/2)

- Recall momentum for point mass:

point mass: m



Linear | rotational

velocity: $v = \dot{r}$, $a = \dot{v} = \ddot{r} \in \mathbb{R}^3$ $\longleftrightarrow$ $\omega = \hat{\omega}\dot{\theta}$, $v = \omega \times r$

force: $f = ma = m\dot{v} = m\ddot{r}$ $\longleftrightarrow$ moment: $n = r \times f$

$a \times b = -b \times a$

Linear momentum: $L = mv$ $\longleftrightarrow$ angular momentum: $\phi = r \times L$

$= r \times (m\, \omega \times r)$

$= m(r \times \omega \times r)$

$= m\, r \times (-r) \times \omega$

$= \boxed{m [r][-r]}\, \omega \rightarrow I$

3x3 matrix

Inertia matrix for this point mass

Rotational Inertia (2/2)

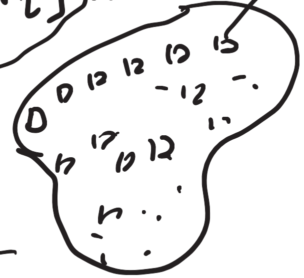
- Rotational Inertia: $\bar{I} \triangleq \int_V \rho(r)[r][r]^T dr$
  - $\rho(\cdot)$ is the density function of the body
  - $\bar{I}$ depends on coordinate system
  - It is a constant matrix if the origin coincides with CoM

density

$\phi = \sum_i m_i [r_i][r_i]^T w$

this matrix depends on coord sys to represent r_i

m_i



$[-r_i] = [r_i]^T$

what's definition for CoM?

$$c \triangleq \text{CoM} = \frac{1}{m}\int \rho(r) \cdot r \, dV$$

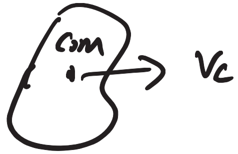
$$c = \frac{1}{m}\sum_i m_i r_i$$

If c is CoM, then

$$\frac{1}{m}\sum m_i(\overrightarrow{cr_i}) = 0$$

$$\boxed{\sum_i m_i \overrightarrow{cr_i} = 0} \Longrightarrow \boxed{\sum_i m_i [\overrightarrow{cr_i}] = 0}$$

Spatial Momentum



- Consider a rigid body with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C
  - Linear momentum: $L = m v_{CoM} = m v_c$

    why? $L = \sum_i m_i v_i = \sum_i m_i (v_c + \omega \times \vec{cr_i})$

    $= (\sum_i m_i) v_c + (\sum_i m_i (\vec{cr_i})) \times \omega$

    $= m v_c$ $\qquad = -\sum_i m_i [\vec{cr_i}] \omega = 0$

  - Angular momentum about CoM:

    $\phi_c = \bar{I}_c \omega$; $\phi_c = \sum_i \vec{cr_i} \times m v_i = \sum_i \vec{cr_i} \times (m_i v_c + m_i \omega \times \vec{cr_i})$

    $= \underbrace{(\sum_i \vec{cr_i} \times m_i v_c)}_{0} + \sum_i m_i \vec{cr_i} \times \omega \times \vec{cr_i}$

    $= \bar{I}_c \omega$

  - Angular momentum about a point O:

    $\phi_o = \sum_i \vec{Or_i} \times (m_i v_i) = \phi_c + \vec{Oc} \times L$

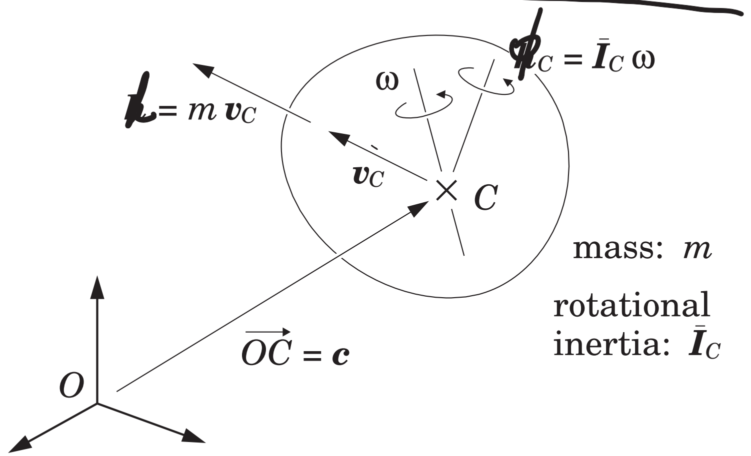
    HW

- Spatial Momentum:

  $$\boxed{\phi_o = \phi_c + \vec{Oc} \times L}$$

  coordinate free

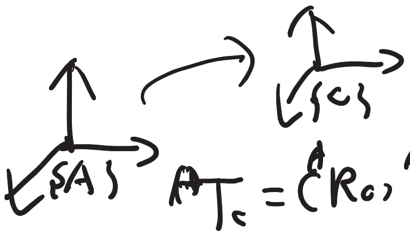
$$h = \begin{bmatrix} \phi \\ L \end{bmatrix} \in \mathbb{R}^6$$



Change Reference Frame for Momentum

- Spatial momentum transforms in the same way as spatial forces:

$$^Ah = {}^AX_C^*\,{}^Ch$$



$^AT_C = ({}^AR_C, {}^Ap_C)$

$$^Ch = \begin{bmatrix} ^C\phi_{O_C} \\ ^CL \end{bmatrix}$$

$$\underbrace{^AL}_{3\times 1} = \underbrace{^AR_C}_{3\times 3}\,\underbrace{^CL}_{3\times 1}$$

$$^Ah = \begin{bmatrix} ^A\phi_{O_A} \\ ^AL \end{bmatrix}$$

$\Leftrightarrow$ Coordinate free

$$\underbrace{\phi_{O_A} = \phi_{O_C} + (\overrightarrow{O_AO_C} \times L)}$$

$\{A\} \Longrightarrow$

$$^A\phi_{O_A} = {}^AR_C\,{}^C\phi_{O_C} + {}^AR_C(-{}^Cp_A \times {}^CL)$$

$$= \begin{bmatrix} ^AR_C & ^AR_C[-{}^Ap_C] \end{bmatrix} \begin{bmatrix} ^C\phi_{O_C} \\ ^CL \end{bmatrix} \Longrightarrow \boxed{^Ah = {}^AX_C^*\,{}^Ch}$$

$\in \mathbb{R}^6$

Spatial Inertia

think about inertia matrix as mapping

$M \longrightarrow F$

- Inertia of a rigid body defines linear relationship between velocity and momentum.

$\mathcal{V}$ twist

wrench

momentum

$\mathcal{H}$

- Spatial inertia $\mathcal{I}$ is the one such that

$$\underset{6\times 1}{h} = \overset{6\times 6}{\mathcal{I}}\,\underset{6\times 1}{\mathcal{V}}$$

- Let $\{C\}$ be a frame whose origin coincide with CoM. Then

$$\overset{6\times 6}{{}^C\mathcal{I}} = \begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

3x3 identity

${}^C\mathcal{I}$

In this case, we know

${}^C\mathcal{V} = \begin{bmatrix} {}^C\omega \\ {}^Cv_c \end{bmatrix}$, ${}^Cv_{com} = {}^Cv_c \Rightarrow {}^CL = m\,{}^Cv_c$

${}^C\phi_c = {}^C\bar{I}_c\,{}^C\omega$

$${}^Ch = \begin{bmatrix} {}^C\bar{I}_c\,{}^C\omega \\ m\,{}^Cv_c \end{bmatrix} = \begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & m\,I_{3\times 3} \end{bmatrix} {}^C\mathcal{V}$$

3x3 identity matrix

Spatial Inertia

$M \to F$

$\downarrow$

- Spatial inertia wrt another frame $\{A\}$:

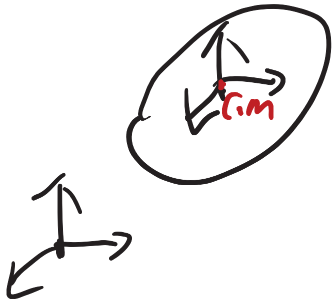
$$\boxed{{}^A\mathcal{I} = {}^AX_C^{*}\,{}^C\mathcal{I}\,{}^CX_A}$$

$$\underline{{}^Ah} = {}^A\mathcal{I}\,{}^A\mathcal{V} = {}^AX_C^{*}\,{}^Ch = {}^AX_C^{*}\,{}^C\mathcal{I}\,{}^C\mathcal{V} = {}^AX_C^{*}\,{}^C\mathcal{I}\,{}^CX_A\,{}^A\mathcal{V}$$

when $\{A\}$ has same orientation of CoM frame ${}^C\mathcal{I}_C$

- Special case: ${}^AR_C = I_3$

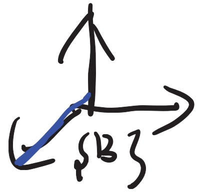
we know ${}^AX_C = \begin{bmatrix} I_3 & 0 \\ [{}^Ap_C] & I_3 \end{bmatrix} \Longrightarrow$



$${}^A\mathcal{I} = \begin{bmatrix} {}^C\mathcal{I} + m[{}^Ap_C][{}^Ap_C]^T & m[{}^Ap_C] \\ m[{}^Ap_C] & m I_{3\times3} \end{bmatrix}$$

Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors



$$\mathcal{F} = \overbrace{\left[e^*_{B_1}\ \ e^*_{B_2}\ \ \cdots\ \ e^*_{B_6}\right]}^{X_B^*}\,{}^B\mathcal{F} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix}$$

force space basis vector

elementary "force/torque"

$$\dot{\mathcal{F}} = \dot{X}_B^*\,{}^B\mathcal{F} + X_B^*\,\underbrace{({}^B\mathcal{F})'}_{{}^B\mathring{\mathcal{F}}}$$

It turns out (if {B} has velocity v_B)

Cross Product for Spatial Force and Momentum

Assume frame A is moving with velocity ${}^A\mathcal{V}_A$

- $\underbrace{{}^A\left[\frac{d}{dt}\mathcal{F}\right]}_{\text{coordinate-free}} = \frac{d}{dt}\left({}^A\mathcal{F}\right) + {}^A\mathcal{V}_A \times^* {}^A\mathcal{F}$

  ${}^A\mathring{\mathcal{F}}$: apparent

- ${}^A\left[\frac{d}{dt}h\right] = \frac{d}{dt}\left({}^Ah\right) + {}^A\mathcal{V} \times^* {}^Ah$

Fact: $[\mathcal{V}\times^*] = -[\mathcal{V}\times]^T$

$$\dot{\mathcal{F}} = X_B^* \, {}^B\mathring{\mathcal{F}} + [\mathcal{V}_B \times^*] X_B^* \, {}^B\mathcal{F}$$

then: $\dot{e}^*_{B_i} = \mathcal{V}_B \times^* e^*_{B_i}$

where "$\times^*$" defined as

$$\mathcal{V} = \begin{bmatrix}\omega \\ v\end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} n \\ f\end{bmatrix}$$

$$\mathcal{V}\times^* \mathcal{F} \triangleq \begin{bmatrix}\omega \times n + v \times f \\ \omega \times f\end{bmatrix}$$

or equivalently

$$[\mathcal{V}\times^*] = \begin{bmatrix}[\omega] & [v] \\ 0 & [\omega]\end{bmatrix}_{6\times 6}$$

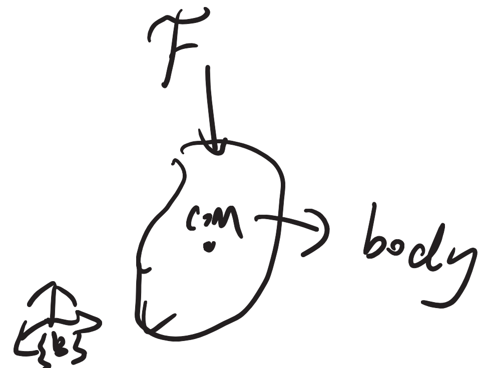
$$\Rightarrow \dot{X}_B^* = [\mathcal{V}_B \times^*] X_B^*$$

Newton-Euler Equation

- Newton-Euler equation:

net external wrench

$$\boxed{\mathcal{F} = \frac{d}{dt}h = \boxed{\mathcal{I}\mathcal{A} + \mathcal{V}\times^* \mathcal{I}\mathcal{V}}}$$



- Adopting spatial vectors, the Newton-Euler equation has the same form in any frame

- ${}^B\mathcal{F} = {}^B\mathcal{I}\,{}^B\mathcal{A} + {}^B\mathcal{V}\times^* {}^B\mathcal{I}\,{}^B\mathcal{V}$ (even when $\{B\}$ is away from CoM)

- ${}^B(\mathcal{F}) = {}^B\left(\frac{d}{dt}h\right) = {}^B\left(\frac{d}{dt}(\mathcal{I}\mathcal{V})\right) = {}^B\left(\mathcal{I}\mathcal{A} + \dot{\mathcal{I}}\mathcal{V}\right)$

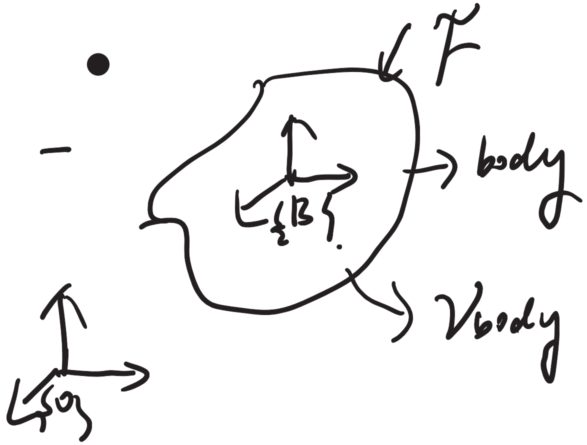
Let's work with inertia frame $\{o\}$ to derive the NE-equation.

$= \mathcal{I}\mathcal{A} + \mathcal{V}\times^*\mathcal{I}\mathcal{V}$

due to $\mathcal{V}$ (velocity) is changing

Accounts for the fact that inertia is moving

Derivations of Newton-Euler Equation

- 

- Assume: {B} attached to the body $\Rightarrow \mathcal{V}_B = \mathcal{V}_{body}$, ${}^B\mathcal{I}$ is constant

- $$\frac{d}{dt}({}^o h) = \frac{d}{dt}({}^o\mathcal{I}\,{}^o\mathcal{V}) = {}^o\dot{\mathcal{I}}\,{}^o\mathcal{V} + {}^o\mathcal{I}\,{}^o\mathcal{A}$$

$$= \frac{d}{dt}\left({}^oX_B^*\,{}^B\mathcal{I}\,{}^BX_o\right){}^o\mathcal{V} + {}^o\mathcal{I}\,{}^o\mathcal{A}$$

$$= {}^o\dot{X}_B^*\,{}^B\mathcal{I}\,{}^BX_o\,{}^o\mathcal{V} + {}^oX_B^*\,{}^B\mathcal{I}\,{}^B\dot{X}_o\,{}^o\mathcal{V} + {}^o\mathcal{I}\,{}^o\mathcal{A}$$

${}^o\mathcal{V} = \mathcal{V}_B$

$$= [{}^o\mathcal{V}_B \times^*]\,{}^oX_B^*\,{}^B\mathcal{I}\,{}^BX_o\,{}^o\mathcal{V} \cancel{- {}^oX_B^*\,{}^B\mathcal{I}\,{}^BX_o\,[{}^o\mathcal{V}_B\times]\,{}^o\mathcal{V}} + {}^o\mathcal{I}\,{}^o\mathcal{A}$$

$$= \boxed{[{}^o\mathcal{V}_B \times^*]\,{}^o\mathcal{I}\,{}^o\mathcal{V} + {}^o\mathcal{I}\,{}^o\mathcal{A}}$$

$$= {}^o\mathcal{I}\,{}^o\mathcal{A} + {}^o\mathcal{V}_B \times^*\,{}^o\mathcal{I}\,{}^o\mathcal{V}$$

Note: ${}^o\dot{X}_B = [{}^o\mathcal{V}_B\times]\,{}^oX_B$, $({}^oX_B\,{}^BX_o) = \mathcal{I} \Rightarrow {}^o\dot{X}_B\,{}^BX_o + {}^oX_B\,{}^B\dot{X}_o = 0 \Rightarrow {}^B\dot{X}_o = -{}^BX_o\,{}^o\dot{X}_B\,{}^BX_o = -{}^BX_o\,[{}^o\mathcal{V}_B\times]$

More Discussions

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More Discussions

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