MEE5114(Sp22) Advanced Control for Robotics

Lecture 1: Linear Differential Equations and Matrix Exponential

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Outline

- Linear System Model

- Matrix Exponential

- Solution to Linear Differential Equations

Motivations

- Most engineering systems (including most robotic systems) are modeled by Ordinary Differential (or Difference) Equations (ODEs)

- Example: Dynamics of 2R robot

differential equation in θ

$$\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta,\dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})},$$

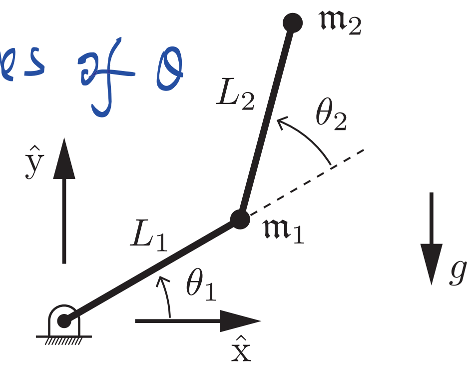
θ 2nd-derivatives of θ

with

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix},$$



- Screw theory, exponential coordinate, and Product of Exponential (PoE) are based on the (linear) differential equation view of robot kinematics

Linear Differential Equations (Autonomous)

- Linear Differential Equations: ODEs that are linear wrt variables e.g.:

e.g. 1 $\begin{cases} \dot{x}_1(t) + x_2(t) = 0 \\ \dot{x}_2(t) + x_1(t) + x_2(t) = 0 \end{cases}$

two - coupled 1^{st}-order ODE involve $x_1(t)$, $x_2(t)$

Vector form: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ in $\mathbb{R}^2$

$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Rightarrow \dot{x} = A \cdot x$

→ A

② $\begin{cases} \ddot{y}(t) + z(t) = 0 \\ \dot{z}(t) + y(t) = 0 \end{cases}$ 2 - variables

$\Rightarrow x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, $x_3(t) = z(t)$

$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$\dot{x} = A \quad x$

- State-space form (1st-order ODE with vector variables):

Linear $\dot{x} = A x$

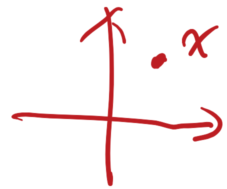
← vector field

General Linear Control Systems

$\Rightarrow$ if $f(x) = Ax$

- General (Autonomous) Dynamical Systems: $\dot{x}(t) = f(x(t))$ $\Rightarrow$ Linear sys $\dot{x} = Ax$
  - $x(t) \in \mathbb{R}^n$: state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$: *vector field*

"Autonomous" means "f" does not depend on non-"x" variables



- Non-autonomous: $\dot{x}(t) = f(x(t), t)$

$\dot{x} = Ax + 2t$, $\dot{x} = Ax + d(t)$ ↑ captures all non-"x" dependence

$\dot{x} = Ax + b$

- Control Systems: $\dot{x}(t) = f(x(t), u(t))$
  - vector field $f : \mathbb{R}^n \times \mathbb{R}^m$ depends on external variable $u(t) \in \mathbb{R}^m$

e.g. $\dot{x} = \underbrace{Ax + \sin(u)}_{f(x,u)}$

- General Linear Control Systems: $\Rightarrow$ $f(x,u) = Ax + Bu$

a $\Leftarrow$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$\Leftarrow$ static relation

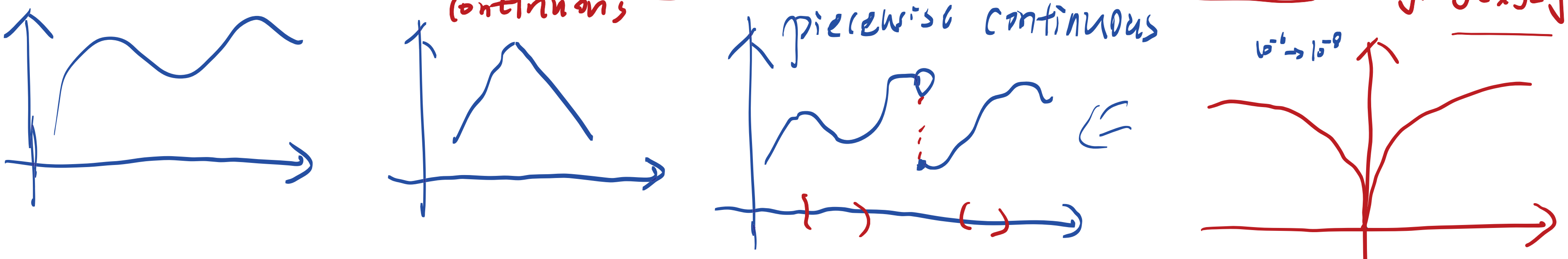
  - $x \in \mathbb{R}^n$: system state, $u \in \mathbb{R}^m$: control input, $y \in \mathbb{R}^p$: system output
  - A, B, C, D are constant matrices with appropriate dimensions

Existence and Uniqueness of ODE Solutions

- Function $g : \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* over domain $\mathcal{D} \subseteq \mathbb{R}^n$ if $\exists L < \infty$ ← exists

$$\Rightarrow \boxed{\|g(x) - g(x')\| \leq L\|x - x'\|, \forall x, x' \in \mathcal{D}}$$

e.g. $g(x) = \sqrt{|x|}$



- **Theorem [Existence & Uniqueness]** Nonlinear ODE

$$① \Leftrightarrow \quad \dot{x}(t) = f(x(t), t), \quad \text{I.C. } x(t_0) = x_0$$

initial condition

has a *unique* solution if $f(x, t)$ is Lipschitz in x and piecewise continuous in t

$$\|f(x,t) - f(x',t)\| \leq L\|x - x'\|, \quad \forall t \in [t_0, t_f]$$

· solution to ① means: $\begin{cases} \langle 1 \rangle : \text{I.C. } x(t_0) = x_0 \\ \langle 2 \rangle \quad \dot{x}(t) = f(x(t), t), \ \forall t \end{cases}$

Existence and Uniqueness of Linear Systems

- **Corollary**: Linear system

$$\hat{f}(x,t) \triangleq Ax + Bu(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad \text{I.C. } x(0) = x_0$$

has a unique solution for any piecewise continuous input $u(t)$

(P.C.)

Proof: check condition:

① $\| \hat{f}(x,t) - \hat{f}(x',t) \| = \| A(x-x') \| \le \underbrace{\|A\|}_{L} \|x-x'\|$

② $\hat{f}(x,t) = (A\underset{\text{fixed}}{x} + Bu(t))$ is also piecewise continuous in t (because $u(t)$ is P.C. in t)

- *Homework*: Suppose A becomes time-varying $A(t)$, can you derive conditions to ensure existence and uniqueness of $\dot{x}(t) = A(t)x(t) + Bu(t)$?

Outline

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

How to Solve Linear Differential Equations?

- General linear ODE: $\dot{x}(t) = Ax(t) + d(t)$ $\in$ $d(t) = Bu(t)$

- The key is to derive solutions to the autonomous linear case: $(\dot{x}(t) = Ax(t))$, with $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and initial condition (IC) $x(0) = x_0$. $\in \mathbb{R}^n$

  n-by-n matrix

- By existence and uniqueness theorem, the ODE $\dot{x} = Ax$ admits a unique solution.

- It turns out that the solution can be found analytically via the *Matrix Exponential*

What is the "Euler's Number" e?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution: $z(t) = (e^{at} \cdot z_0)$

proof: ① check I.C. $z(0) = e^{0} \cdot z_0 = z_0$ ✓

② check vector field: $(e^{at} \cdot z_0)' = a \cdot (e^{at} \cdot z_0)$

$\dot{z}(t) = a \cdot z(t)$ ✓

- What is the number "e"?

- Euler's number

- Defined as the number such that $(e^x)' = e^x$ | $(2^x)' \neq 2^x$

$\Rightarrow \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \Rightarrow \frac{e^h - 1}{h} \xrightarrow{h \to 0} 1$

$\Rightarrow e^h \to h+1 \Rightarrow e^h = \lim_{h \to 0} (h+1)^{1/h}$

$\Rightarrow e = 2.71..$

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x \triangleq \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$(e^x)' = 1 + x + \frac{x^2}{2!} + \cdots$

$= e^x$

$x \in$

- This can be extended to complex variables:

$f(z)$

$$e^z \triangleq \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

This power series is well defined for all $z \in \mathbb{C}$

- In particular, we have $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots$

Let $z = j\cdot\theta$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

$$\begin{cases} \sin\theta = \boxed{-\frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}} \\ \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \end{cases} \Rightarrow \underbrace{e^{j\theta} = \cos\theta + j\sin\theta}_{\text{Euler's formula}}$$

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = \left(\underbrace{I}_{n\times n} + \underbrace{A}_{n\times n} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)$$

E.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A^3 = 0$ . . . .

$$e^A = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{bmatrix} + \cdots$

- This power series is well defined for any finite square matrix $A \in \mathbb{R}^{n\times n}$.

Some Important Properties of Matrix Exponential

① • $Ae^A = e^A A$ proof: ~~Def~~ By definition: $e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$

$Ae^A = A(\sum \frac{A^i}{i!}) =$

But: $Ae^B \neq e^B A$, if $AB \neq BA$

② • $e^A e^B = e^{A+B}$ if $AB = BA$ ← see next page

③ • If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$ (P is nonsingular)

A is similar to D

$e^A = I + PDP^{-1} + \frac{PD\cancel{P^{-1}}\cancel{P}DP^{-1}}{2!} + \cdots$

$\frac{PD^2P^{-1}}{2!} + \cdots$

$= Pe^D P^{-1}$

④ • For every $t, \tau \in \mathbb{R}$, $e^{At}e^{A\tau} = e^{A(t+\tau)}$

From ②: $\underbrace{(At)}_{A_1} \cdot \underbrace{(A\tau)}_{A_2} = A_2 A_1$

⑤ • $(e^A)^{-1} = e^{-A}$

From ②: $e^A \cdot e^{-A} = e^{A+(-A)} = e^0 = I$

Outline

proof of ②:

$$e^{A+B} = \sum_{N=0}^{\infty} \frac{(A+B)^N}{N!} \overset{?}{=} \left(\sum_{N=0}^{\infty} \frac{A^N}{N!}\right)\left(\sum \frac{B^N}{N!}\right)$$

~~proof~~ property:

$$\boxed{\binom{N}{k} = \frac{N!}{k!(N-k)!}}$$

$$= \sum_{N=0}^{\infty} \frac{\sum_{k=0}^{N} \binom{N}{k} A^k B^{N-k}}{N!}$$

holds if $AB = BA$

$$= \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} \frac{A^k B^{N-k}}{k!(N-k)!} =$$

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

Autonomous Linear Systems

$x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.
- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$\boxed{x(t) = e^{At}x_0} \Leftarrow \text{func of } t$$

Proof: ① check I.C. $x(0) = e^{0} \cdot x_0 = I \cdot x_0 = x_0$ ✓

② check vector field: $\underbrace{\frac{d}{dt}(e^{At}x_0) \stackrel{?}{=} A \cdot (e^{At}x_0)}_{\text{we need to show this}}$

By definition: $\frac{d}{dt}(e^{At} \cdot x_0) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right) \cdot x_0$

$= \left(A + A^2t + A^3\frac{t^2}{2!} + \cdots\right)x_0 = A\underbrace{\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)}_{x_0}$

Computation of Matrix Exponential (1/2) $= A \cdot e^{At} \cdot x_0$ ✓

- Directly from definition : $e^{At} = \underline{\sum_{k=0}^{\infty} \frac{(At)^k}{k!}}$
  - It's hard to compute
  - For special case, this series have analytical form

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdots \Rightarrow e^{At} = I + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + 0 \cdots$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

- For diagonalizable matrix:

Example: $A = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$

$$\Rightarrow At = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{P^{-1}}$$

$$\Rightarrow \text{By property } 3, \quad e^{At} = P \cdot e^{D \cdot t} P^{-1} = P \cdot \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{2t}, & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Computation of Matrix Exponential (2/2)

- Using Laplace transform

$$\hat{X}(s) = \int x(t) e^{-st} dt$$

- $\dot{x} = Ax$, $x(0) = x_0 \in \mathbb{R}^n$

Laplace transform:

$$\boxed{\begin{aligned} x(t) &\Longleftrightarrow \hat{X}(s) \in \mathbb{R}^n \\ \dot{x}(t) &\Longleftrightarrow s\hat{X}(s) - x(0) \end{aligned}}$$

Apply $s\hat{X}(s) - x(0) = A\hat{X}(s) \Rightarrow (sI - A)\hat{X}(s) = x_0$

$$\Rightarrow \hat{X}(s) = (sI - A)^{-1} x_0$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1} x_0\right]$$

we also know $x(t) = e^{At} x_0 \Rightarrow \boxed{e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)}$

Solution to General Linear Systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3}$$

- $x \in \mathbb{R}^n$ is system state, $u \in \mathbb{R}^m$ is control input, $y \in \mathbb{R}^p$ is the system output

- A, B, C, D are constant matrices with appropriate dimensions

- **Homework:** The solution to the linear system (3) is given by

$$\begin{cases} x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{cases}$$

More Discussions